Journal of Computer and Mathematical Sciences, *Vol.9(5), 391-397 May 2018* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Fixed Point Theorems in Menger Space Satisfying Occasionally Weakly Compatible Mappings

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(Received on: April 7, 2018)

ABSTRACT

In this present paper we have utilize the notion of occasionally weakly compatibility to prove the results in Menger Space which is more suitable framework in many situations.

Keywords: Menger Space, Occasionally Weakly Compatible (OWC).

1. INTRODUCTION

Metric space can be generalized. One such generalization is Menger Space introduced by Menger[4] who used distribution functions instead of nonnegative real numbers as values of the metric, the notion of probabilistic metric space correspond to situation when we do not know exactly the distance between the two points but we know probabilities of possible values of this distance. A probabilistic generalization of metric space appears to be interest in the investigation of physical quantities and physiological threshold. It is also a fundamental importance in probabilistic functional analysis. Schweizer and Sklar[7] studied this concept and then the important development of Menger Space theory was due to Sehgal[9] and Bharucha-Reid[1]. The basic result of PM- space introduced by Schweizer and Sklar[7,8]. Then Sehgal[9] initiated the study of contraction mapping theorems in PM-spaces. Subsequently, several contraction mapping theorems for commuting mappings have been proved in PM-spaces; see for instance Ciric *et al*.[2], Pant and Kumar[5].

In this paper we have proved some fixed point results in Menger Spaces satisfying an Occasionally Weakly Compatible mapping which is an extension of the recent work of Shukla.M, Dubey.R, Patel.S.K.

The study of fixed points for non-compatible mappings can be extended to the class of non-expansive or Lipschitz mapping pairs even without continuity of the mappings involved.

Besides the above we choose to utilize the notion of occasionally weakly compatible to prove our results in Menger Space, which is a wider and suitable framework in many situations. Here one may observe that we need not improve the completeness requirements of the space or the containment of the ranges of the involved mappings. Our result complement comparable results in the literature Ciric *et al*.[3], Sedghi *et al*.[10] & Singh *et al*.[12].
Before going to our main result we require some more definitions and lemma,

Definition 1.1: Let X be a non empty set and L denote the set of all distribution functions. A probabilistic metric space is an ordered pair (X,F) where $F : X * X \rightarrow L$. we shall denote the distribution function by F (p, q) or $F_{p,q}$; $p, q \in X$ and F(p, q, x) will represent the value of F (p, q) at $x \in R$. the function $F_{p,q}$ is assumed to satisfy the following conditions :

1. $F_{p,q}(t) = 1 \ \forall t > 0$ if and if $p = q$
2. $F_{p,q}(0) = 0$ for every $p, q \in X$
3. $F_{p,q}(t) = F_{q,p}(t)$ for every $p, q \in X$
4. If $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$ it follows that $F_{q\,r}(t+s) = 1 \ \forall p, q, r \in X$ and $t, s \geq 0$.

In metric space (X ,d) , the metric d induces a mapping $F : X * X \rightarrow L$ such that $F_{p,q}(t) = H(t - d(p, q))$ for all $p, q \in X$ and $t \in R$, where H is the distribution function defined as

$$H(x) = \begin{cases} 0, \text{ if } x \leq 0 \\ 1, \text{ if } x > 0 \end{cases}$$

Definition 1.2: A mapping $\Delta: [0, 1] * [0,1] \rightarrow [0,1]$ is called t- norm if the following conditions are satisfied

(1) $\Delta(a, 1) = a$ for all $a \in [0, 1]$, $\Delta(0,0) = 0$,
(2) $\Delta(a, b) = \Delta(b, a)$
(3) $\Delta(c, d) \leq \Delta(a, b)$ for $c \geq a$, $d \geq b$, and
(4) $\Delta(\Delta(c, d), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c \in [0,1]$

Definition 1.3: A Menger probabilistic space is a triplet (X, F, Δ) where (X, F) is a PM-space and Δ is a t- norm with the following condition
$F_{p,r}(t+s) \geq \Delta(F_{p,r}(t), F_{p,r}(s))$ for all $p, q, r \in X$ and $t, s \geq 0$.
The above inequality is called Menger's triangle inequality.

Definition 1.4: A sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent to a point $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n, X}(\varepsilon) \rightarrow 1 - \lambda \ \forall n \geq N(\varepsilon, \lambda)$.

Definition 1.5: A sequence $\{x_n\}$ in (X, F, Δ) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n, x_m}(\varepsilon) \rightarrow 1 - \lambda \ \forall n, m \geq N(\varepsilon, \lambda)$.

Definition 1.6: A Menger Space (X, F, Δ) with the continuous t- norm is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.7: let A Menger Space (X, F, Δ) .Two mapping $f, g : X \rightarrow X$ are said to be compatible if and only if $F_{fg x_n, gf x_n}(t) \rightarrow 1$ for all $t > 0$ whenever $\{x_n\}$ in X such that $fx_n, g x_n \rightarrow z$ for some $z \in X$.

Definition 1.8: Two self mapping f and g of a probabilistic metric space (X,F,Δ) are said to be point wise R- weakly commuting if given $x \in X$, there exists $R > 0$ such that
$F_{fgx,\, gfx}(t) \geq F_{fx,\, gx}(t/R)$ for $t > 0$.
And T and R-weak commutative of type (A_g) if $F_{ffx,\, gfx}(t) \geq F_{fx,\, gx}(t/R)$

Definition 1.9: Let $\{x_n\}$ be a sequence in PM space (X, F, Δ).The self map S and T are called non – compatible if there exist at least one sequence $\{x_n\}$ in X such that $\lim_n Tx_n = \lim_n Sx_n = p$ for some p in X but $\lim_n F_{STx_n, TSx_n}(t) \neq 1$ or $\lim_n F_{STx_n, TSx_n}(t)$ does not exists.

Definition 1.10: Two self maps f and g of a set X are Occasionally Weakly Compatible (OWC) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 1.11: A function $\emptyset : [0,\infty)$ is said to be a $\emptyset$- function if it satisfies the following conditions:

I. $\emptyset(t) = 0$ if and only if $t = 0$,
II. $\emptyset(t)$ is strictly increasing and $\emptyset(t) \to \infty$ as $t \to \infty$
III. $\emptyset(t)$ is left continuous in$(0,\infty)$ and
IV. $\emptyset$ is continuous at 0,

An altering distance function with the additional property that $h(t) \to \infty$ as $t \to \infty$ generates a $\emptyset$ function in the following way:

$$\emptyset(t) = \begin{cases} \sup\{s : h(s) < t \text{ if } t > 0 \\ 0 \quad \text{if } t = 0 \end{cases}$$

It can be easily seen that $\emptyset$ is a function $\emptyset$- function.

Lemma 1.1: Let $(X.F, \Delta)$ be a Menger Space, A and B are (OWC) self maps of X. If A and B have a unique point of coincidence, w = Ax = Bx, then w is the unique common fixed point of A and B.

Proof: since A and B are (OWC), there exists a point $x \in X$ such that Ax = Bx = w and ABx = BAx. Thus, AAx = ABx = BAx, which says that Ax is also a point of coincidence of A and B. since the point of coincidence w = Ax is unique by hypothesis, BAx = AAx = Ax, and w = Ax is a common fixed point of A and B. Moreover, if z is any common fixed point of A and B, then
Z = Az = Bz = w by the uniqueness of the point of coincidence.
Now we are in position to produce the following results.

2. MAIN RESULTS

Theorem 2.1: Let $(X.F, \Delta)$ be a complete Menger Space and let p, q, f and g be self mapping of X. Let pairs {p, f} and {q, g} be occasionally weakly compatible (OWC). If there exists $k \in (0, 1)$ such that

$$F_{px,qy}(kt) \geq \min\left\{\begin{array}{c} F_{fw,gy}(t), F_{px,fx}(t), F_{qy,gy}(t), F_{px,gy}(t), F_{fx,qy}(t), \\ \frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2} \end{array}\right\} \ldots\ldots..2.1.1$$

For all x, y ϵ X and for all t > 0, then there exists a unique point w ϵ X such that pw = fw = w and a unique point z ϵ X such that qz = gz = z. Moreover, z = w, so that there is a unique common fixed point of p, f, q and g.

Proof: Since the pairs {p, f} and {q, g} be (OWC), there exists point x, y ϵ X such that px = fx and qy = gy. We claim that px = qy. If not, by inequality 2.1.1

$$F_{px,qy}(kt) \geq \min\left\{\begin{matrix} F_{fx,gy}(t), F_{px,fx}(t), F_{qy,gy}(t), F_{px,gy}(t), F_{fx,qy}(t), \\ \frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} F_{px,qy}(t), F_{px,px}(t), F_{qy,qy}(t), F_{px,qy}(t), F_{px,qy}(t), \\ \frac{F_{qy,qy}(t)+F_{px,px}(t)}{2}, \frac{F_{px,qy}(t)+F_{px,qy}(t)}{2} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} F_{px,qy}(t), 1,1, F_{px,qy}(t), F_{px,qy}(t), \\ \frac{1+1}{2}, \frac{2F_{px,qy}(t)}{2} \end{matrix}\right\}$$

$$= F_{px,\,qy}(t)$$

Thus we have px = qy. Therefore, px = qy = fx = gy.

Suppose that there is another point z such that pz = fz then 2.1.1 we have pz = fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f.

By lemma 1.1 w is the only common fixed point of p and f. Similarly there is a unique point Z ϵ X such that z = qz = gz

Uniqueness: assume that w ≠ z and taking x= w, y = z inequality (2.1.1) we get $F_{wz}(kt) = F_{pw,qz}(kt)$

$$\geq \min\left\{\begin{matrix} F_{fw,gz}(t), F_{pw,fw}(t), F_{qz,gz}(t), F_{pw,gz}(t), F_{fwqz}(t), \\ \frac{F_{qz,gz}(t)+F_{fw,pw}(t)}{2}, \frac{F_{fw,gz}(t)+F_{pw,gz}(t)}{2} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} F_{w,z}(t), F_{w,w}(t), F_{z,z}(t), F_{w,z}(t), F_{w,z}(t), \\ \frac{F_{z,z}(t)+F_{w,w}(t)}{2}, \frac{F_{w,z}(t)+F_{w,z}(t)}{2} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} F_{w,z}(t), 1,1, F_{w,z}(t), F_{w,z}(t), \\ \frac{1+1}{2}, \frac{2F_{w,z}(t)}{2} \end{matrix}\right\}$$

$$= F_{w,z}(t)$$

Therefore we have z = w by lemma (1.1) and z is a common fixed point of p, f, q and g.

The uniqueness of the fixed point holds from (2.1.1)

Theorem 2.2: Let (X, F, Δ) be a complete Menger Space and let p , q , f and g be self mapping of X. Let pairs {p, f} and {q, g} be (OWC). If there exists k ϵ (0, 1) such that

$$F_{px,qy}(kt) \geq \emptyset \left[\min\left\{\begin{matrix} F_{fx,gy}(t), F_{px,fx}(t), F_{qy,gy}(t), F_{px,gy}(t), F_{fx,qy}(t), \\ \frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2} \end{matrix}\right\}\right]$$

For all x, y ϵ and $\emptyset \in \varphi$ for all $0 < t < 1$, then there exists a unique common fixed point of p, f, q and g.

Proof: The proof follows from theorem 2.1.

Theorem 2.3: Let (X, F, Δ) be a complete Menger Space and let p , q , f and g be self mapping of X. Let pairs {p, f} and {q, g} be (OWC). If there exists k ϵ (0, 1) such that

$$F_{px,qy}(kt) \geq \emptyset\left[\min\left\{\begin{matrix} F_{fx,gy}(t), F_{px,fx}(t), F_{qy,gy}(t), F_{px,gy}(t), F_{fx,qy}(t), \\ \frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2} \end{matrix}\right\}\right] \quad ....2.3.1$$

For all x ,y ϵ X and $\emptyset: [0,1]^7 \to [0,1]$ such that $\emptyset(t,1,1,t,t,1,t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of p , f , q and g.

Proof: Since the pairs {p, f} and {q, g} are (OWC)**,** there exists points x, y ϵ X X such that px = fx and qy = gy. We claim that px = qy. If not, by inequality 2.3.1 we have

$$F_{px,qy}(kt) \geq \emptyset\left[\min\left\{\begin{matrix} F_{fx,gy}(t), F_{px,fx}(t), F_{qy,gy}(t), F_{px,gy}(t), F_{fx,qy}(t), \\ \frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2} \end{matrix}\right\}\right]$$

$$= \emptyset\left\{\begin{matrix} F_{px,qy}(t), F_{px,px}(t), F_{qy,qy}(t), F_{px,qy}(t), F_{px,qy}(t), \\ \frac{F_{qy,qy}(t)+F_{px,px}(t)}{2}, \frac{F_{px,qy}(t)+F_{px,qy}(t)}{2} \end{matrix}\right\}$$

$$= \emptyset\left\{\begin{matrix} F_{px,qy}(t), 1, 1, F_{px,qy}(t), F_{px,qy}(t), \\ F_{px,qy}(t), 1, F_{px,qy}(t) \end{matrix}\right\} > F_{px,qy}(t)$$

This contradiction, thus, we have px = qy. Therefore, px = qy = fx = gy.

Suppose that there is another point z such that pz = fz then by (2.3.1) we have pz = fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f.

By Lemma (1.1) w is the only common fixed point of p and f. Similarly there is a unique point z ϵ X such the z = qz = gz. Thus z is a common fixed point of p,f,q and g. The uniqueness of the fixed point holds from (2.3.1)

Theorem 2.4 Let (X, F, Δ) be a complete Menger Space with continuous t-norm Δ and $\Delta(t, t) \geq t$ for all t ϵ [0, 1] and let p, q, f and g be self mapping of X. Let pairs {p, f} and {q, g} be OWC. If there exists k ϵ (0, 1) such that

$$F_{px,qy}(kt) \geq \begin{matrix} \Delta(\Delta F_{fx,gy}(t), \Delta(F_{px,fx}(t), \Delta F_{qy,gy}(t), \Delta F_{px,gy}(t), \Delta F_{fx,qy}(t), \\ \Delta(\frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2}))))) \end{matrix} \quad .....2.4.1$$

for all x, y ϵ X and t > 0, then there exists a unique common fixed point of
p, f, q, and g.

Proof: Since the pairs {p, f} and {q, g} are (OWC), there exists points x, y, ϵ X such that px = fx and qy = gy. We claim the px = qy. By inequality 2.4.1 we have

$$F_{px,qy}(kt) \geq \begin{matrix} \Delta(\Delta F_{fx,gy}(t), \Delta(F_{px,fx}(t), \Delta F_{qy,gy}(t), \Delta F_{px,gy}(t), \Delta F_{fx,qy}(t), \\ \Delta(\frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2}))))) \end{matrix} \quad .....2.4.1$$

$$F_{px,qy}(kt) \geq \begin{array}{c} \Delta(\Delta F_{px,qy}(t), \Delta(F_{px,px}(t), \Delta F_{qy,qy}(t), \Delta F_{px,gy}(t), \Delta F_{px,qy}(t), \\ \Delta(\frac{F_{qy,qy}(t)+F_{px,px}(t)}{2}, \frac{F_{px,qy}(t)+F_{px,qy}(t)}{2}))))) \ldots\ldots \end{array}$$

$$F_{px,qy}(kt) \geq \begin{array}{c} \Delta(\Delta F_{px,qy}(t), \Delta(1, \Delta(1, \Delta(F_{px,gy}(t), \Delta F_{px,qy}(t), \\ \Delta(1, \frac{F_{px,qy}(t)+F_{px,qy}(t)}{2}))))) \ldots\ldots \end{array}$$

$$> F_{px,qy}(kt)$$

Thus, we have px = qy, Therefore, px = qy = fx = gy.

Uniquesness

Suppose that there is another point z such the pz = fz then by 2.4.1 we have pz = fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f.
By Lemma 1.1 w is the only common fixed point of p and f. Similarly there is a unique $z \in X$ such that z = qz = gz. Thus w is a common fixed point of p, f, q, and g. The uniqueness of the fixed point holds from 2.4.1.

3. CONCLUSION

Occasionally Weakly Compatible (OWC) mappings always commute at coincidence point of mappings and we have find this coincidence point as common fixed point of mappings. These OWC mappings are very helpful to find common fixed point of mappings under any kind of contractive conditions in Menger Space whether it is rational expression or general.

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