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Energy and Spectrum of an Undirected Graph $G_{m,n}$

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ABSTRACT

In this paper the notation of $G_{m,n}$ be a basic simple undirected graph with vertex set $V = I_n = \{1,2,3,\cdots,n\}$ and $u, v \in V$ are adjacent if and only if $u \neq v$ and $u + v$ is not divisible by m, where $m \in N$ and $m > 1$. We have determined the energies and spectrum of the graph $G_{m,n}$.

AMS Subject Classification: 05C50, 05C35.

Keywords: Energy of a graph, Spectrum of a graph, Matrix energy of a graph.

1. INTRODUCTION

The concept of Energy of a Graph was introduced by I. Gutman[1] in 1978. A great variety of graph energies is being considered in the current mathematical chemistry. It can be used to approximate the total π-electron energy of a molecule[2]. This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. Now a day's graph energy is referred to as, closely related to the total π-electron energy calculated within the Huckel molecular orbital approximation[3].

Let G be a graph with n vertices and m edges and the adjacency matrix of $A(G)$ of G is defined by its entries as $a_{ij} = 1$, if two vertices are adjacent and 0 otherwise. Let the eigen values of $A(G)$ be $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$.Then spectral radius λ_1 is the highest eigen value of G. Then we will write $\lambda_i(G)$ instead of λ_i.We know $\det A = \prod_{i=1}^{n} \lambda_i$. Spectrum is the collection of Eigen values with their multiplicities

$(m_1, m_2, \dots, m_n)$ of an adjacency matrix $A(G)$. If at least one of its eigen value is zero, the graph G is said to be singular and for singular graph $\det A = 0$. All eigen values are different from zero then the graph is non-singular, Then $\det A > 0$. The energy of G is defined to be the sum of absolute values of the eigen values of G and it is denoted by $\mathcal{E}(G)$. i.e., $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, and it is extensively studied by D. Cvetkovia and X.Li.Y.Shi[4,5].

Nikiforov[6] generalised the matrix energy of any graph is defined as the sum of singular values of the adjacency matrix of G, and it is denoted by $\mathcal{E}_m(G)$.

2. THE UNDIRECTED GRAPH ON A FINITE SUBSET OF NATURAL NUMBERS AND ITS PROPERTIES

Definition 2.1: Ivy. Chakrabarty[7] introduced an undirected graph $G_{m,n}$ on a finite subset of natural numbers and proved some basic properties of $G_{m,n}$. Let $G_{m,n}$ be a simple undirected graph with vertex set $V = I_n = \{1,2,3,\cdots,n\}$ and $u, v \in V$ are adjacent if and only if $u \neq v$ and $u + v$ is not divisible by m, where $m \in N$ and $m > 1$. If $m = 1$, the graph is disconnected and it forms only isolated vertices. Some of the properties of $G_{m,n}$ are

Lemma 2.1: Let $m, n \in N,\ m, n > 1$. Then the graph $G_{m,n}$ is connected.

Lemma 2.2: $G_{m,n} \cong K_3$ if and only if $n = 3$ and $m \geq 6$.

Lemma 2.3: $G_{m,n}$ is a $(n-2)$-regular graph for $n = m - 1$, where m is odd.

Lemma 2.4: $G_{m,n}$ is a complete k −partite graph, if $n = m - 1$, where m is odd and $k = \frac{n}{2}$.

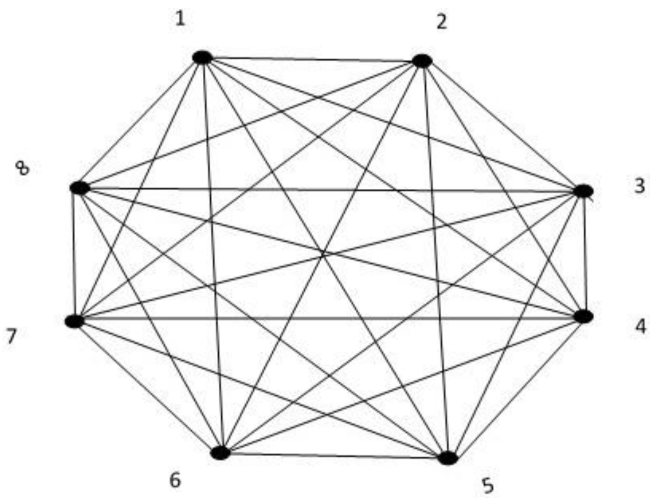
Lemma 2.5: Let $m \geq 2n$.Then $G_{m,n}$ is complete.

3. ENERGY AND SPECTRUM OF A GRAPH $G_{m,n}$:

Definition 3.1: Let $G_{m,n}$ be an undirected graph with n vertices and m be the positive integer > 1 and let $A = (a_{ij})$ be the adjacency matrix of $A(G)$ of G is defined by its entries as $a_{ij} = 1$, if two vertices are adjacent and 0 otherwise and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ are the eigen values of $A(G)$. The spectrum of $G_{m,n}$ is $\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m_1 & m_2 & \dots & m_n \end{pmatrix}$ and the energy is the sum of the absolute values of the eigen values of $G_{m,n}$. i.e.,$\mathcal{E}(G_{m,n}) = \sum_{i=1}^{n} |\lambda_i|$.

Theorem 3.1: The energy of the graph $G_{m,n}$ is $2n - 2$ if $n \geq 3$ and $m \geq 2n$.

Proof: Let $V = \{1,2,3,\dots,n\}$ be the vertex set of the graph $G_{m,n}$ where $m, n \in N \ and \ m > 1$. If $n \geq 3$ and $m \geq 2n$, then the graph $G_{m,n}$ is complete and connected.



For the graph, $G_{16,8}$, $\varepsilon(G_{16,8}) = 14$

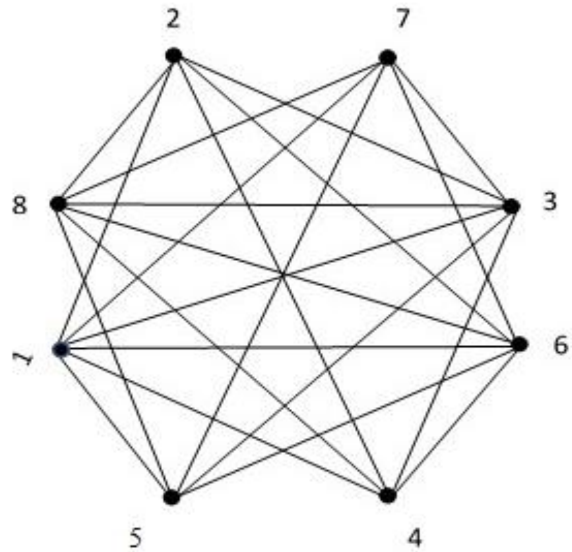
The adjacency matrix of $G_{m,n}$ graph is an $n \times n$ matrix defined as $A(G_{m,n}) = (a_{ij}) = \begin{cases} 1, & if\ \text{u}_\text{i} \text{ and } \text{v}_\text{j} \text{ are adjacent}, \\ 0, & otherwise. \end{cases}$

Then $A(G_{m,n}) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{n \times n}$.

The characteristic equation is $|A - \lambda I| = 0$.
This implies $(\lambda + 1)^{(n-1)} + [\lambda - (n-1)] = 0$.
Therefore $Spec(G_{m,n}) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$.
Hence the energy of a graph is $\mathcal{E}(G_{m,n}) = \sum_{i=1}^{n} |\lambda_i| = \sum_{j=1}^{2} |\lambda_j|\, spec(\lambda_j)$
$= |\lambda_1| spec(\lambda_1) + |\lambda_2| spec(\lambda_2) = 2n - 2$.

Theorem 3.2: The energy of the graph $G_{m,n}$ is $2n - 4$ **i**f $n = m - 1$ where m is odd and ≥ 5.

Proof: Let $n = m - 1$ where m is odd and ≥ 5 .
Let $V = \{m-1, m-2, \dots, 2, 1\}$ be the vertex set of $G_{m,n}$.
Consider two subsets of V as $V_i = \{m - i, i\}$ and $V_j = \{m - j, j\}$.
Then the vertex $m - i$ is adjacent to $m - j$,
Because $(m - i) + (m - j)$ is not divisible by m as $(m - j) \neq$ i.



For the graph, $G_{9,8}$, $\varepsilon(G_{9,8}) = 12$

The adjacency matrix of the graph $G_{m,n}$ is $A(G_{m,n}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 0 & 1 & \vdots & \vdots & 1 & 0 & 1 \\ 1 & 1 & \ddots & 1 & 1 & ⋰ & 1 & 1 \\ 1 & \dots & 1 & 0 & 0 & 1 & 1 & 1 \\ \vdots & \dots & 1 & 0 & 0 & 1 & \dots & \vdots \\ 1 & 1 & ⋰ & 1 & 1 & \ddots & 1 & 1 \\ 1 & 0 & 1 & \vdots & \vdots & 1 & 0 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 0 \end{pmatrix}_{n\times n}$

and the characteristic equation is $(\lambda + 2)^{\left(\frac{n}{2}-1\right)} + (\lambda)^{\frac{n}{2}} + \left(\lambda - (n-2)\right) = 0$ and

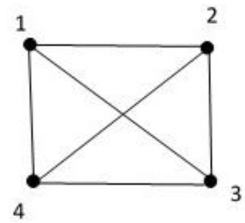
$Spec(G_{m,n}) = \begin{pmatrix} -2 & 0 & n-2 \\ \frac{n}{2}-1 & \frac{n}{2} & 1 \end{pmatrix}$.

Hence the energy $\mathcal{E}(G_{m,n}) = 2n - 4$.

4. MATRIX ENERGY AND SPECTRUM OF A GRAPH $\mathcal{E}_{\boldsymbol{m}}(\boldsymbol{G})$:

Definition 4.1: Let A(G) be the adjacency matrix of G and A(G)$'$ be the transpose of A(G). Then A(G)A(G)$'$is a positive semi definite matrix and the eigen values and singular values of the G are same. The matrix energy of G is denoted by $\mathcal{E}_m(G)$ and is defined as the summation of singular values of A(G).so, the energies $\mathcal{E}(G)$ and $\mathcal{E}_m$(G) both are same.

Theorem 4.1: The matrix energy of $G_{m,n}$ is $2n - 2$ if $n \geq 3$ and $m \geq 2n$.
Proof:



For the graph, $G_{8,4}$, $\varepsilon_m(G_{8,4}) = 6$

Then $A(G_{m,n})A'(G_{m,n}) = \begin{pmatrix} n-1 & n-2 & n-2 & \dots & n-2 \\ n-2 & n-1 & n-2 & \dots & n-2 \\ n-2 & n-2 & n-1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & n-2 & \dots & n-1 \end{pmatrix}_{n\times n}$

and the characteristic equation is $(\lambda - 1)^{n-1} + [\lambda - (n-1)] = 0$.

Therefore $Spec(G_{m,n}) = \begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix}$ and the singular values are $1, n-1$.

Hence the matrix energy of $G_{m,n} = \mathcal{E}_m(G_{m,n})$ =summation of singular values of $A(G_{m,n}) = \sum_{j=1}^{2}|\lambda_j|\, spec(\lambda_j) = |\lambda_1|spec(\lambda_1) + |\lambda_2|spec(\lambda_2) = 2n-2$.

Theorem 4.2: The matrix energy of the graph $G_{m,n}$ is $2n-4$ if $n = m-1$ where m is odd and ≥ 5.

Proof: From theorem 3.2, we have

$$A(G_{m,n}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 0 & 1 & \vdots & \vdots & 1 & 0 & 1 \\ 1 & 1 & \ddots & 1 & 1 & \iddots & 1 & 1 \\ 1 & \dots & 1 & 0 & 0 & 1 & 1 & 1 \\ \vdots & \dots & 1 & 0 & 0 & 1 & \dots & \vdots \\ 1 & 1 & \iddots & 1 & 1 & \ddots & 1 & 1 \\ 1 & 0 & 1 & \vdots & \vdots & 1 & 0 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 0 \end{pmatrix}_{n\times n}.$$

Then $A(G_{m,n})A'(G_{m,n}) =$

$$\begin{pmatrix} n-2 & n-4 & n-4 & n-4 & n-4 & \dots & n-4 & n-2 \\ n-4 & n-2 & n-4 & \vdots & \vdots & n-4 & n-2 & n-4 \\ n-4 & n-4 & \ddots & n-4 & n-4 & \iddots & n-4 & n-4 \\ n-4 & \dots & n-4 & n-2 & n-2 & n-4 & n-4 & n-4 \\ \vdots & \dots & n-4 & n-2 & n-2 & n-4 & \dots & \vdots \\ n-4 & n-4 & \iddots & n-4 & n-4 & \ddots & n-4 & n-4 \\ n-4 & n-2 & n-4 & \vdots & \vdots & n-4 & n-2 & n-4 \\ n-2 & n-4 & n-4 & \dots & \dots & n-4 & n-4 & n-2 \end{pmatrix}_{n\times n}$$

and the Characteristic equation is $(\lambda)^{\left(n-\frac{n}{2}\right)} + (\lambda - 2)^{\left(n-\left(\frac{n}{2}+1\right)\right)} + \left(\lambda - (n-2)\right) = 0.$

And the $Spec(G_{m,n}) = \begin{pmatrix} 0 & 2 & n-2 \\ n-\frac{n}{2} & n-\left(\frac{n}{2}+1\right) & 1 \end{pmatrix}$.

Hence the matrix energy of $G_{m,n} = \mathcal{E}_m(G_{m,n}) = 2n - 4.$

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