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Ideals in $(m, m+1)$-semilattice

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ABSTRACT

A commutative semigroup S is called an (m, n) $-$semilattice if $x^n = x^m$, for all $x \in S$, where m and n are positive integers with $m < n$. The relation $\leq$ is defined as $a \leq b$ if $a = ab$, $\forall\ a, b \in S$. In this paper, we prove that every maximal filter is prime in a prime $(m, m+1)$-semilattice and we characterize prime $(m, m+1)$-semilattice by Stone's Theorem and Extended Ideals.

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1. INTRODUCTION

The notion of ideal is a fundamental concept in the modern theory of semigroups. The theory of ideals in semigroups was studied by[3.1,2] and[13] had given some characterizations of prime down-sets and maximal filters in semilattices.[14] had given a condition for the existence of least upper bound of elements in a Poset. The class of weakly distributive semilattices was introduced by[9] with the name prime semilattices and were intensively studied in[4,8] and [10,9]

introduced the prime semilattice as the semilattice S satisfying any one of the following equivalent conditions:
(i) If $x_1 \vee x_2 \vee \dots \dots \vee x_n$ exists in S, then for each x in S,
$(x \wedge x_1) \vee (x \wedge x_2) \dots \dots (x \wedge x_n)$ exists and equals $x \wedge (x_1 \vee x_2 \vee \dots \dots \vee x_n)$.
(ii) If F is a filter in S and J a non empty subset of S disjoint with F and such that $x_1 \vee x_2 \vee \dots \dots \vee x_n$ exists, whenever $x_1, x_2, \dots \dots x_n \in J$, then there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \varphi$.
(iii) If $x \nleq y$, then there exists a prime filter F' such that $x \in F'$ and $y \notin F'$.

This theorem is commonly known as Balbes Theorem.

A modified version of the Balbes Theorem was obtained by[10,9] have proved that every maximal filter is prime in a prime semilattice. Ideals in prime semilattices were studied by[8] and[10]. They had obtained several characterizations of prime semilattices by extended ideals.

A commutative semigroup S is called an (m,n) −semilattice if $x^n = x^m$, for all $x \in S$, where m and n are positive integers with $m < n$.We have studied (m,n) −semilattice in[11]. In[12], we have introduced a relation $\leq$ as $a \leq b$ if $a = ab$, $\forall\ a, b \in S$, which is anti-symmetric and transitive. Using this relation we have defined the filter F as a non empty subset of a commutative semigroup S such that $ab \in F$, $\forall\ a, b \in F$ and $a \in F$ with $a \leq b$ implies that $b \in F$.

Now, we collect some known results used in the sequel. We have proved that if S is an (m,n) −semilattice and $a \in S$ then $[a) = B = \{b \in S / a^m \leq b\}$ is a filter called the principal filter of S. Also, if F is a filter of an $(m, m+1)$-semilattice S and $a \notin F$ then we have obtained the filter generated by $F \cup \{a\}$ as the set $\{b \in S / f a^m \leq b$, for some $f \in F\}$.

In this paper, we study ideals of an $(m, m+1)$-semilattice. First section contains some basic results about ideals of a commutative semigroup. Second section deals with order ideals of a commutative semigroup S. We prove that an order ideal is an ideal of S. In the third section, we show that the complement of a maximal filter is a minimal prime down-set. Fourth section deals with ideals of an $(m, m+1)$-semilattice S and we obtain a condition for the existence of least upper bound of elements of S. In the next section we define prime filter of an $(m, m+1)$ - semilattice S and using this we introduce the prime $(m, m+1)$ - semilattice. And we show that prime $(m, m+1)$-semilattices are characterized by Stone's Theorem. In the concluding section, we have another characterization for prime $(m, m+1)$-semilattice using extended ideals. For the semilattice theoretic concepts which have now become commonplace, the reader is referred to[5 and 6] and for semigroup theory to[7].

2. IDEALS OF A COMMUTATIVE SEMIGROUP

In this section, we give some results related to ideals of a commutative semigroup.

Theorem 2.1. Let S be an (m,n) −semilattice, $m \geq 2$ and $a \in S$. Then
(i) $I_a = \{x \in S / x^m \leq a^{n-m}\}$ is an ideal of the commutative semigroup S, containing a. And $B_a \subseteq < a > \subseteq I_a$, where $B_a = \{x \in S / x \leq a\}$ and $< a >$ is the smallest ideal of a commutative semigroup containing a.

(ii) $I_a \cap B_s \subseteq B_{a^{n-m}} \subseteq I_a$, where $B_s = \{a^l \ / \ a \in S\}$ is the semilattice of an (m,n) −semilattice S such that $m \leq l < n$ and $l \equiv 0(mod(n - m))$
(iii) $I_a \cap I_b \subseteq I_{ab}$ and $I_a \subseteq I_{a^k} \forall \ k \geq 2$,

Proof (i) If $x \in I_a$ and $y \in S$ then $x^m \leq a^{n-m}$ and hence by Theorem 5 (i) of [12] $(xy)^m \leq a^{n-m}$which implies that $xy \in I_a$. Again by Theorem 5 (i) of [12], $a^m \leq a^{n-m} \Longrightarrow a \in I_a$ $i.e. < a > \subseteq I_a$ and by Theorem 1 (xi) of [12], $B_a \subseteq < a >$.
(ii) If $x \in I_a \cap B_s$ then $x^m \leq a^{n-m}$ and $x = x^2$ which implies that $x \leq a^{n-m}$ and hence $x \in B_{a^{n-m}}$. Again if $x \in B_{a^{n-m}}$. Then $x \leq a^{n-m}$ which implies that $x^m \leq a^{n-m}$and hence x$\in I_a$.
(iii) If $x \in I_a \cap I_b$ then $x^m \leq a^{n-m}$ and $x^m \leq b^{n-m}$ and hence by Theorem 1 of [12] $x^m \leq (ab)^{n-m}$. Therefore, $x \in I_{ab}$. Again, if $x \in I_a$then $x^m \leq a^{n-m}$ and hence by Theorem 5 (v) of [12], $x^m \leq a^{k(n-m)}$, $\forall \ k \geq 2$, which implies that $x \in I_{a^k}$, $\forall \ k \geq 2$.

Theorem 2.2. Let S be an (m,n) −semilattice and $n - m$ divides m then
(i) $I_a \cap I_b$= I_{ab} and I_a =I_{a^k}, $\forall \ k \geq 2$. (ii) $I_a = I_b$ if and only if a^m = b^m. i.e. a$\sim_m b$.

Proof (i) If $x \in I_{ab}$ then $x^m \leq (ab)^{n-m}$, by Theorem 5 (v) of [12], $x^m \leq (ab)^m$ and hence by Theorem 5(ii) of [12], $x^m \leq a^{n-m}$ and $x^m \leq b^{n-m}$ which implies that $x \in I_a$ and $x \in I_b \Longrightarrow x \in I_a \cap I_b$. Again, by Theorem 2.1 (iii), $I_a \cap I_b = I_{ab}$. If $x \in I_{a^k}$ then $x^m \leq a^{k(n-m)}$. By Theorem 5 (v) of [12], $x^m \leq a^{km}$and since a^m is idempotent, $x^m \leq a^m$, then by Theorem 5 (i) of [12], $x^m \leq a^{n-m}$ which shows that $x \in I_a$. Again by Theorem 2.1 (iii), I_a =I_{a^k} . (ii) By Theorem 5 (i) of [12], $a \in I_a$ and $b \in I_b$. Suppose $I_a = I_b$ then $a \in I_b$ and $b \in I_a$. i.e. $a^m \leq b^{n-m}$ and $b^m \leq a^{n-m}$and hence by Remark 8 of [12], $a^m \leq b^m$and $b^m \leq a^m$ which implies that $a^m = b^m$. Therefore, by Lemma 40 of [11], a$\sim_m$b . Conversely, suppose $a^m = b^m$ then $x \in I_a$ if and only if $x^m \leq a^{n-m}$, i. e. by Remark 8 of [12], if and only if $x^m \leq a^m$. i.e. if and only if $x^m \leq b^m$ i.e. if and only if $x^m \leq b^{n-m}$. i.e. if and only if $x \in I_b$ and hence $I_a = I_b$

3. ORDER IDEAL

In this section, we introduce an ideal of the commutative semigroup S, using the relation ≤ and we call it as an order ideal of S.

Definition 3.1. Let S be a commutative semigroup. A non empty subset D of S is called a down-set if $a \in D$, $x \in S$ and $x \leq a$ then $x \in D$.

A proper down-set D of S is called a prime down-set if $a, b \in S$ and $ab \in D$ then either $a \in D$ or $b \in D$.

A non empty subset D of S is said to be up-directed if for $x, y \in D$, $\exists \ z \in D$ such that $x \leq z$ and $y \leq z$. If D is up-directed then for $x \in D$, $\exists z \in D$ such that $x \leq z$.

An up-directed down-set of S is called an order ideal of S.

Remark 3.2. An ideal of a commutative semigroup S is a down-set of S.

Proof Suppose, $x \in D$ and $y \in S$ be such that $y \leq x$, then $y = yx \in D$.

Remark 3.3. An order ideal of a commutative semigroup S is an ideal of S.

Proof For, suppose I is an order ideal of S. Let $x \in I$ and $s \in S$. Then $\exists\ z \in I$ such that $x \leq z$. Hence, $sx \leq z$ and hence, $sx \in I$. So, I is an ideal of the commutative semigroup S.

Remark 3.4. An ideal of a commutative semigroup S need not be an order ideal.
For example, consider the (3,4)-semilattice S={a,b,c,d} with $\circ$ defined as follows.

∘	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	a	a	c
d	a	c	c	b

D = {a,b,c} is an ideal of S and hence a down-set of S. But, D is not up-directed, since we have $a \leq d$ and $c \leq d$, but $d \notin D$ and hence D is not an order ideal of S.

Lemma 3.5. Finite intersection of order ideals of a commutative semigroup S is an order ideal of S.

Proof Let $I_1, I_2, \ldots \ldots I_n$ be order ideals of S and let $x_i \in I_i, \forall\ i = 1,2, \ldots \ldots, n$. Then by Remark 3.3, $x_1 x_2 \ldots \ldots x_n \in I_i, \forall\ i = 1,2, \ldots \ldots, n$. Hence I$=\bigcap_{i=1}^{n} I_i \neq \varphi$.
If $a \in I$ and $x \leq a$ then $a \in I_i, \forall\ i = 1,2, \ldots \ldots, n$. Hence $x \in I_i, \forall\ i = 1,2, \ldots \ldots, n$. i.e. $x \in I$.Let $x, y \in I$ then $\exists\ z_i \in I_i$ such that $x \leq z_i$, $y \leq z_i$, $\forall\ i = 1,2, \ldots \ldots, n$.. Hence $x \leq z$ and $y \leq z$, where $z = z_1 z_2 \ldots \ldots z_n$. Since $z_i \in I_i$ and I_i is an ideal of S, $z \in I_i$, $\forall\ i = 1,2, \ldots \ldots, n$. i.e. $z \in I$. Therefore I is an order ideal of S.

Proposition 3.6. If S is a commutative semigroup and $a \in S$ is an idempotent then $B_a = \{x \in S / x \leq a\}$ = < a > is an order ideal of S.

Proof Since a, is an idempotent, $a \leq a$ and hence $a \in B_a$. If $x \in B_a$ and $y \leq x$ then $y \leq a$. i.e. $y \in B_a$. If $x, y \in B_a$ then $x \leq a, y \leq a$ and $a \in B_a$. Hence B_a is an order ideal of S.

Corollary 3.7. Let S be an (m, n)-semilattice then (i) if $n \geq 2m$ then $B_{a^{n-m}}$ is an order ideal of S, $\forall\ a \in S$ and (ii) if $n < 2m$ and $n - m$ divides m then $B_{a^{n-m}}$ is an order ideal of S, $\forall\ a \in S$.

Proof If $n \geq 2m$, then by Note 27 of [11],for any $a \in S$, a^{n-m} is an idempotent and hence by Proposition 3.6, $B_{a^{n-m}}$ is an order ideal of S. If $n < 2m$ and $n - m$ divides m then by Remark 8 of [12], $B_{a^{n-m}} = B_{a^m}$ and by Note 27 of [11], a^m is an idempotent. Hence $B_{a^{n-m}}$ is an order ideal of S.

Note 3.8. In particular, if $n = m + 1$, then $B_a = B_{a^m}$ is an order ideal of an (m, n)-semilattice S, $\forall\ a \in S$.

Remark 3.9. If S is an (m,n)-semilattice and $0 \in S$, then 0 is in any order ideal of S.

4. MAXIMAL FILTER

In this section, we define maximal filter of an $(m, m+1) -$ semilattice and study about its complement.

Definition 4.1. A maximal filter of an $(m, m+1)$-semilattice S is a proper filter which is not contained in any other proper filter, that is, if there is a proper filter F' such that F$\subseteq F'$, then F$= F'$.

Theorem 4.2. Let F be a non empty subset of a commutative semigroup S. If F is a proper filter then $S - F$ is a prime down-set and if F is a prime down-set then $S - F$ is a proper filter.

Proof Suppose F is a proper filter. Then $S - F \neq \varphi$ and $S - F \neq S$. Let $a \in S - F$ and $b \in S$ with $b \leq a$. If $b \in F$ then $a \in F$, which is a contradiction. Hence $b \in S - F$ and hence $S - F$ is a proper down-set.
Let $xy \in S - F$. Suppose $x \in F$ and $y \in F$ then $xy \in F$, which is a contradiction. Hence either $x \in S - F$ or $y \in S - F$. Therefore $S - F$ is a prime down-set.
Suppose F is a prime down-set. If $x, y \in S - F$, then $xy \notin F$. i.e. $xy \in S - F$. If $a \in S - F$ and $b \in S$ with $a \leq b$. Suppose, $b \in F$, then $a \in F$, which is a contradiction. Hence, $b \in S - F$ and hence $S - F$ is a filter.

Following Lemmas are needed for further development of the paper.

Lemma 4.3. Every proper filter of a commutative semigroup S is contained in a maximal filter.

Proof Let F be a proper filter of S. Let $\mathcal{P}$ be the set of all proper filters containing F. Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Let $M = \cup \{X / X \in \mathcal{C}\}$. We claim that M is a filter such that $M \supseteq F$. Since by Remark 14 of [12], Union of any chain of filters is a filter, M is a filter and $M \supseteq F$. Thus M is the maximal element of $\mathcal{C}$. Therefore, by Zorn's Lemma, $\mathcal{P}$ has a maximal element F' with $F' \supseteq F$.

Lemma 4.4. Every prime down-set of a commutative semigroup S contains a minimal prime down-set.

Proof Let D be a prime down-set of S. Let $\mathcal{P}$ be the set of all prime down-sets contained in D. Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Let $M = \cap \{X / X \in \mathcal{C}\}$. We claim that M is a prime down-set such that $M \subseteq D$. Suppose $a \in M$ and $b \in S$ be such that $b \leq a$. Then $a \in X$, for all $X \in \mathcal{C}$ which implies that $b \in M$.
Also, if $ab \in M$ then $ab \in X, \forall X \in C$, which implies that either $a \in M$ or $b \in M$. Hence, M is a prime down-set and $M \subseteq D$. Thus M is the minimal element of $\mathcal{C}$. Therefore, by the Dual form of Zorn's Lemma, $\mathcal{P}$ has a minimal element D' with $D' \subseteq D$.

Corollary 4.5. If F is a maximal filter of a commutative semigroup S, then $S - F$ is a minimal prime down-set and if F is a minimal prime down-set , then $S - F$ is a maximal filter.

Proof Suppose F is a maximal filter of S, then by Theorem 4.2, $S - F$ is a prime down-set of S. Suppose $S - F$ is not a minimal prime down-set. Then by Lemma 4.4, there exists a minimal prime down-set F' such that $S - F \supseteq F'$ which implies that F$\subseteq S - F'$, which is a contradiction, since F is a maximal filter. Hence, $S - F$ is a minimal prime down-set.
Suppose F is a minimal prime down-set of S, then by Theorem 4.2, $S - F$ is a proper filter of S. Suppose $S - F$ is not a maximal filter. Then by Lemma 4.3, there exists a maximal filter F' such that $S - F \subseteq F'$ which implies that F$\supseteq S - F'$, which is a contradiction, since F is a minimal prime down-set. Hence, $S - F$ is a maximal filter.

5. IDEAL

In this section, we introduce another ideal of an $(m, m + 1)$-semilattice, depending upon the existence of the least upper bound of elements of an $(m, m + 1)$ −semilattice.

Notation 5.1. Let S be an $(m, m + 1)$-semilattice. The least upper bound of $\{x_1, x_2, \dots \dots, x_n\} \subseteq S$, if it exists is denoted by $x_1 + x_2 + \cdots + x_n$. Clearly, $x_1 + x_2 + \cdots + x_n$ is an idempotent.
Next we give a condition for the existence of least upper bound of elements of an $(m, m + 1)$ −semilattice.

Lemma 5.2. In an $(m, m + 1)$ −semilattice S, for $x_1, x_2, \dots \dots, x_n \in S$, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists if and only if $[x_1) \cap [x_2) \cap \dots \cap [x_n)$ is a principal filter. Also, whenever ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$exists, $[x_1) \cap [x_2) \cap \dots \cap [x_n)$ =$[{x_1}^m + {x_2}^m + \cdots + {x_n}^m)$.

Proof Suppose, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists. Then, for $z \in [{x_1}^m + {x_2}^m + \cdots + {x_n}^m)$ we have, $({x_1}^m + {x_2}^m + \cdots + {x_n}^m)^m \leq z$. Also ${x_i}^m \leq {x_1}^m + {x_2}^m + \cdots + {x_n}^m$, $\forall\, i = 1,2, \dots \dots, n$. which implies that ${x_i}^m \leq ({x_1}^m + {x_2}^m + \cdots + {x_n}^m)^m$,$\forall\, i = 1,2, \dots \dots, n$ and hence ${x_i}^m \leq z$,$\forall\, i = 1,2, \dots \dots, n$.Thus, $z \in [x_i), \forall\, i = 1,2, \dots \dots, n$.
i.e. $z \in [x_1) \cap [x_2) \cap \dots \cap [x_n)$.
Again, if $z \in [x_1) \cap [x_2) \cap \dots \cap [x_n)$, then ${x_i}^m \leq z$,$\forall\, i = 1,2, \dots \dots, n$. i. e. z is an upper bound of $\{{x_1}^m, {x_2}^m, \dots, {x_n}^m\}$. Hence ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \leq z$ which implies that $z \in [{x_1}^m + {x_2}^m + \cdots + {x_n}^m)$.
Suppose, $[x_1) \cap [x_2) \cap \dots \cap [x_n)$ is a principal filter. Then, $[x_1) \cap [x_2) \cap \dots \cap [x_n) = [y)$. Since $y \in [y)$, $y \in [x_i)$,$\forall\, i = 1,2, \dots \dots, n$, which implies ${x_i}^m \leq y, \forall\, i = 1,2, \dots \dots, n$. i. e. ${x_i}^m \leq y^m, \forall\, i = 1,2, \dots \dots, n$. Hence, y^m is an upper bound of $\{{x_1}^m, {x_2}^m, \dots, {x_n}^m\}$. Suppose, there exists another upper bound b, then ${x_i}^m \leq b, \forall\, i = 1,2, \dots \dots, n$.Therefore, $b \in [x_i)$,$\forall\, i = 1,2, \dots \dots, n$. i. e. $b \in [x_1) \cap [x_2) \cap \dots \cap [x_n)$. i. e. $b \in [y)$ and hence, $y^m \leq b$. Thus, y^m is the least upper bound of $\{{x_1}^m, {x_2}^m, \dots, {x_n}^m\}$ and hence the least upper bound exists. But the least upper bound of $\{{x_1}^m, {x_2}^m, \dots, {x_n}^m\}$, if it exists is ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$. Therefore, $y^m = {x_1}^m + {x_2}^m + \cdots + {x_n}^m$.
Now we give the definition of an ideal of an $(m, m + 1)$ −semilattice.

Definition 5.3. Let S be an $(m, m+1)$-semilattice. We call a down-set D of S as an ideal of S, if ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$exists in S, for ${x_1}^m, {x_2}^m, \dots, {x_n}^m \in D$,
then ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in D$.
We call an ideal D of S as a prime ideal if the down-set D of S is prime.

Remark 5.4. An order ideal of an $(m, m+\ 1)$ −semilattice S is an ideal of S.

Proof Let D be an order ideal of S. Then D is a down-set of S. Suppose ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$exists in S, for ${x_1}^m, {x_2}^m, \dots, {x_n}^m \in D$. Since, D is up-directed, there exists $z \in D$ such that ${x_i}^m \leq z, \forall\ i = 1,2, \dots \dots, n$. i. e. z is an upper bound of $\{{x_1}^m, {x_2}^m, \dots, {x_n}^m\}$. Thus, we have, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \leq z$.
Hence, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in D$.

Remark 5.5. An ideal of an $(m, m\ +\ 1)$ −semilattice S need not be an order ideal of S.

For example, consider the $(3, 4)$ −semilattice S=$\{a, b, c, d\}$ with ∘ defined as in the Remark 3.4. D=$\{a, b, c, \}$ is a down-set of S. And a is the least upper bound of =$\{a^3, b^3, c^3, \}$. Also, $a \in D$ and hence D is an ideal of S.Again, for a,$c \in D$, $a \leq d, c \leq d$ holds and there exists no other common upper bounds for $\{a, c, \}$But $d \notin D$. Therefore, D is not up-directed, and hence, D is not an order ideal of S.

Remark 5.6. A semigroup ideal need not be an ideal of an $(m, m+1) -$ semilattice.
For example, consider the $(2,3)$ −semilattice
$S = \{(x, a), (x, b), (x, c), (x, d), (y, a), (y, b), (y, c), (y, d)\}$ with ∘ defined as in the following table.

o	(x, a)	(x, b)	(x, c)	(x,d)	(y, a)	(y, b)	(y, c)	(y, d)
(x, a)	(x, a)	(x, a)	(x, a)	(x, a)	(y, a)	(y, a)	(y, a)	(y, a)
(x, b)	(x, a)	(x, b)	(x, c)	(x, d)	(y, a)	(y, b)	(y, c)	(y, d)
(x, c)	(x, a)	(x, c)	(x, a)	(x, c)	(y, a)	(y, c)	(y, a)	(y, c)
(x, d)	(x, a)	(x, d)	(x, c)	(x, d)	(y, a)	(y, d)	(y, c)	(y, d)
(y, a)	(y, a)	(y, a)	(y, a)	(y, a)	(y, a)	(y, a)	(y, a)	(y, a)
(y, b)	(y, a)	(y, b)	(y, c)	(y, d)	(y, a)	(y, b)	(y, c)	(y, d)
(y, c)	(y, a)	(y, c)	(y, a)	(y, c)	(y, a)	(y, c)	(y, a)	(y, c)
(y, d)	(y, a)	(y, d)	(y, c)	(y, d)	(y, a)	(y, d)	(y, c)	(y, d)

$D = \{(x, a), (y, a), (y, c), (y\ d)\}$ is a semigroup ideal. Therefore, by Remark 3.2, D is a down-set of S. Also, (x, b) and (x, d) are the only common upper bounds of $\{(x, a), (y, a), (y\ d)\}$. But $(x, d) \leq (x, b)$. Hence, (x, d) is the least upper bound of $\{(x, a), (y, a), (y\ d)\}$ and $(x, d) \notin D$. Therefore D is not an ideal of S.

Lemma 5.7. Finite intersection of ideals of an $(m, m\ +\ 1)$ −semilattice S is an ideal of S, provided the intersection is non empty.

Proof Let $I_1, I_2, \dots \dots, I_n$ be ideals of S. Suppose, I=$\bigcap_{i=1}^{n} I_i \neq \varphi$.Let $a \in I$ and $x \in S$ be such that $x \leq a$. then $a \in I_i$, ,$\forall\ i = 1,2, \dots \dots, n$.and hence $x \in I_i$, $\forall\ i = 1,2, \dots \dots, n$. which implies that $x \in I_1 \cap I_2 \cap \dots \dots \cap I_n = I$ and hence I is a down-set of S. Suppose, for ${x_1}^m, {x_2}^m, \dots, {x_n}^m \in I$, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$exists, then ${x_1}^m, {x_2}^m, \dots, {x_n}^m \in I_i$,

$\forall\ i = 1,2, \dots \dots, n$ and hence $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in I_i, \forall\ i = 1,2, \dots \dots, n$. Therefore, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in I_1 \cap I_2 \cap \dots \dots \cap I_n = I$, showing that I is an ideal of S.

Proposition 5.8. Let S be an $(m, m + 1)$ −semilattice and $a \in S$. Then $B_a = \{x \in S / x \le a\}$ is an ideal of S .

Proof follows by Proposition 3.6 and Remark 5.4.

Proposition 5.9. Let S be an $(m, m + 1)$ −semilattice and $y \in$S. Then $J_y = \{x \in S \ / \ x^m \le y\}$ is an ideal of S.

Proof Since, $y^m \le y$, $y \in J_y$. If $x \in J_y$ and $a \in S$ be such that $a \le x$, then by Corollary 7 (i) of [12], $a^m \le a \le x$, and hence, $a^m \le x$, which implies, by Corollary 7 (iii) of [12], $a^m \le x^m$.Thus, $a^m \le y$, and hence, $a \in J_y$. Therefore, J_y is a down-set of S. Suppose, for $x_1{}^m, x_2{}^m, \dots, x_n{}^m \in J_y$ if $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ exists in S, then $(x_i{}^m)^m \le y$, $\forall\ i = 1,2, \dots \dots, n$ and hence, $x_i{}^m \le y$, $\forall\ i = 1,2, \dots \dots, n$. Therefore, y is an upper bound of $\{x_1{}^m, x_2{}^m, \dots, x_n{}^m\}$. But the least upper bound of $\{x_1{}^m, x_2{}^m, \dots, x_n{}^m\}$ is $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$, and hence, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \le y$, which implies, by Corollary 7 (i) of [12], $(x_1{}^m + x_2{}^m + \cdots + x_n{}^m)^m \le y$. $Thus\ \ x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in J_y$ and hence, J_y is an ideal of S.

6. PRIME $(m, m + 1)$ −semilattice

In this section, we introduce and study prime $(m, m + 1)$-semilattice and we prove that every maximal filter is a prime filter in a prime $(m, m + 1)$-semilattice.
To begin with, we define the prime filter of an $(m, m + 1)$ −semilattice.

Definition 6.1. A proper filter F of an $(m, m + 1)$ −semilattice S is said to be prime if whenever $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ exists in S, and is an element of F, for $x_1, x_2 \dots x_n \in S$, then $x_i{}^m \le F$, for some i, $1 \le i \le n$.
The following Theorem, which has certain interest of its own is necessary for the characterization of prime $(m, m + 1)$-semilattice.

Theorem 6.2. Let S be an $(m, m + 1)$-semilattice. If F is a prime filter of S, then $S - F$ is a prime ideal of S and if F is a prime ideal of S then $S - F$ is a prime filter of S.

Proof Let F be a prime filter of S. Then by Theorem 4.2, $S - F$ is a prime down-set of S. Suppose, for $x_1{}^m, x_2{}^m, \dots, x_n{}^m \in S - F$, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$exists in S. Suppose, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in F$, then $x_i{}^m \in F$, for some i, $1 \le i \le n$, which is a contradiction. Hence, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in S - F$, showing that $S - F$ is a prime ideal of S.
Again, if F is a prime ideal of S then by Theorem 4.2, $S - F$ is a proper filter of S. Suppose, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ exists in S and $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in S - F$.
Suppose, $x_i{}^m \in F$,$\forall\ i = 1,2, \dots \dots, n$, then, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in F$, which is a contradiction, and hence, $x_i{}^m \in S - F$, for some i, $1 \le i \le n$. Therefore, $S - F$ is a prime filter of S.

In what follows, we consider the condition for an $(m,m+1)-$semilattice to be a prime $(m,m+1)-$semilattice.

Theorem 6.3. In an $(m,m+1)-$semilattice S, the following are equivalent:
(i) For $x_1, x_2, \dots\dots, x_n \in S$, if ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists in S, then for each $x \in S$, $x^m {x_1}^m + x^m {x_2}^m + \cdots + x^m {x_n}^m$ exists and equals $x^m({x_1}^m + {x_2}^m + \cdots + {x_n}^m)$.
(ii) If F is a filter of S and J is an ideal of S disjoint from F, then there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \varphi$.
(iii) If F is a filter of S and J is a non empty subset of S disjoint with F and such that $x_1 + x_2 + \cdots + x_n$ exists in J whenever $x_1, x_2, \dots, x_n \in J$, then there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \varphi$.
(iv) If $x^m \nleq y$, then there exists a prime filter F' such that $x \in F'$ and $y^m \notin F'$.

Proof (i)⟹(ii)
Let $\mathcal{P}$ be the set of all filters that contains F and disjoint with J. Then $\mathcal{P}$ is partially ordered by inclusion. Since the union of chain of filters in S is a filter in S, by Zorn's Lemma, $\mathcal{P}$ has a maximal element F' . Clearly, F' is a proper filter of S, containing F and $F' \cap J = \varphi$. Suppose, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists in S and ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in F'$. Suppose, ${x_i}^m \notin F'$, $\forall\, i = 1,2, \dots\dots, n$. Then, for each i, let F_i be the filter generated by $F' \cup \{x_i\}$. By the maximality of F' , $F_i \cap J \neq \varphi$. Hence, there exists $j_i \in F_i$ and $j_i \in J$, by Theorem 24 of [12], we have, $f_i {x_i}^m \leq j_i$, for some $f_i \in F'$ and $j_i \in J$ and hence for each i, $f^m {x_i}^m \leq j_i$, where $f = f_1 f_2 \dots f_n \in F'$ and since J is a down-set of S, $f^m {x_i}^m \in J$, $\forall\, i = 1,2, \dots\dots, n$. Again, since S is prime and J is an ideal of S, $f^m {x_1}^m + f^m {x_2}^m + \cdots + f^m {x_n}^m \in J$. Therefore, $f^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \in J$. Also, $f^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \in F'$. Hence, $f^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \in F' \cap J$, which is a contradiction. Therefore, F' is a prime filter of S.
(ii) ⟹ (iii)
Suppose, $x_1, x_2, \dots\dots, x_n \in J$, then ${x_1}^m, {x_2}^m, \dots, {x_n}^m \in J$. Therefore, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists in J and hence J is an ideal of S. Thus, by (ii), there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \varphi$.
(iii) ⟹ (iv)
Suppose, $x^m \nleq y$, then $x^m \nleq y^m$. Hence, $F = [x) = \{b \in S \,/\, x^m \leq b\}$ is a filter of S and $J = \{y^m\}$ is a non empty subset of S such that $F \cap J = \varphi$.
Therefore, by (iii), there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \varphi$. Thus,
$x \in F \subseteq F'$ and $y^m \notin F'$.
(iv) ⟹ (i)
Suppose, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists in S and $x \in S$. Then, $x^m {x_i}^m \leq x^m({x_1}^m + {x_2}^m + \cdots + {x_n}^m)$, for each i. Suppose there exists $w \in S$, such that $x^m {x_i}^m \leq w$, for each i, but $x^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \nleq w$. Since, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ is an idempotent, by (iv), there exists a prime filter F such that $x^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \in$F and $w^m \notin F'$. Also, since, $x^m ({x_1}^m + {x_2}^m + \cdots + {x_n}^m) \leq {x_1}^m + {x_2}^m + \cdots + {x_n}^m$, we have, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in F$. Since F is a prime filter, ${x_i}^m \in F$, for some i, $1 \leq i \leq n$. Again, $x^m ({x_1}^m + {x_2}^m + \cdots +$

${x_n}^m)\leq\ x^m$ implies that $x^m\in F$ and hence $x^m\ {x_i}^m\in F$, for some i, $1\leq i\leq n$ and $w^m\notin F$, which is a contradiction, since $x^m{x_i}^m\leq$w, for each i. Therefore, $x^m\ ({x_1}^m+{x_2}^m+\cdots+{x_n}^m)$ is the least upper bound of $\{x^m{x_1}^m, x^m{x_2}^m, \dots, x^m{x_n}^m\}$. But the least upper bound, if it exists is $x^m{x_1}^m+x^m{x_2}^m+\cdots+x^m{x_n}^m$and hence $x^m{x_1}^m+x^m{x_2}^m+\cdots+x^m{x_n}^m$=$x^m\ ({x_1}^m+{x_2}^m+\cdots+{x_n}^m)$, showing that S is prime.

Definition 6.4. An $(m, m\ +\ 1)$ −semilattice S inwhich one of the equivalent conditions in Theorem 6.3 holds is called a prime $(m, m\ +\ 1)$- semilattice.

Theorem 6.5. In a prime $(m, m\ +\ 1)$-semilattice S, every maximal filter is prime.

Proof Let F be a maximal filter in S. Suppose, for some $n\geq 2$, ${x_1}^m+{x_2}^m+\cdots+{x_n}^m$exists in S and ${x_1}^m+{x_2}^m+\cdots+{x_n}^m\in F$. Suppose, ${x_i}^m$ ∉F, for i=2,3,.........,n. Then for each such i, let F_i be the filter generated by x_i and F is not proper. So, $x_1\in F_i$, and hence by Theorem 24 of [12], $f_i{x_i}^m\leq x_1$, for some $f_i\in F$ and i=2,3,........,n. But then, $(f_1f_2\dots f_n)^m{x_i}^m\leq x_1$, for i=2,3,.........,n. Also, we have, $(f_1f_2\dots f_n)^m{x_1}^m\leq x_1$. Hence, x_1 is an upper bound of $\{(f_1f_2\dots f_n)^m{x_i}^m\}, \forall\ i=1,2,\dots\dots, n$. But the least upper bound of $\{(f_1f_2\dots f_n)^m{x_i}^m\}, \forall\ i=1,2,\dots\dots, n$, if it exists is, $(f_1f_2\dots f_n)^m{x_1}^m$+ $(f_1f_2\dots f_n)^m{x_2}^m$+…………..+$(f_1f_2\dots f_n)^m{x_n}^m$. Therefore, $(f_1f_2\dots f_n)^m{x_1}^m$+ $(f_1f_2\dots f_n)^m{x_2}^m$+…………..+ $(f_1f_2\dots f_n)^m{x_n}^m\leq x_1$which implies that, $(f_1f_2\dots f_n)^m{x_1}^m$+ $(f_1f_2\dots f_n)^m{x_2}^m$+…………..+ $(f_1f_2\dots f_n)^m{x_n}^m\leq {x_1}^m$. Since, S is prime, , $(f_1f_2\dots f_n)^m(\ {x_1}^m+{x_2}^m+\cdots {x_n}^m)\leq {x_1}^m$. Again, since, $(f_1f_2\dots f_n)^m$ $(\ {x_1}^m+{x_2}^m+\cdots+{x_n}^m)\in F$, ${x_1}^m\in F$ and hence F is a prime filter of S.

Lemma 6.6. In a prime $(m, m\ +\ 1)$ −semilattice S, every proper filter is contained in a prime filter.

Proof follows by Lemma 4.3 and Theorem 6.5.
Next we give another characterization for the complement of a maximal filter.

Lemma 6.7. In a prime $(m, m\ +\ 1)$ −semilattice S, the complement of a maximal filter is a minimal prime ideal and the complement of a minimal prime ideal is a maximal filter.

Proof Let F be a maximal filter of S. By Corollary 4.5, $S-F$ is a minimal prime down-set. Suppose, ${x_1}^m+{x_2}^m+\cdots+{x_n}^m$exists for ${x_1}^m, {x_2}^m, \dots, {x_n}^m\in S-F$ and suppose, ${x_1}^m+{x_2}^m+\cdots+{x_n}^m\in F$. Since S is prime, F is a prime filter, and hence, ${x_i}^m\in F$, for some i, which is a contradiction. Thus, ${x_1}^m+{x_2}^m+\cdots+{x_n}^m\in S-F$ and $S-F$ is a minimal prime ideal.
Again, if F is a minimal prime ideal, then by Theorem 4.2, $S-F$ is a proper filter. Suppose $S-F$ is not a maximal filter, then by Lemma 6.6, there exists a prime filter F' such that $S-F\subseteq F'$, which implies that $S-F'\subseteq F$, and by Theorem 6.2, $S-F'$ is a prime ideal, which is a contradiction, since, F is a minimal prime ideal. Therefore, $S-F$ is a maximal filter.

Theorem 6.8. Let S be a prime $(m, m\ +\ 1)$ −semilattice and $y\in S$. If $x\notin J_y$,
then there exists a prime ideal containing J_y and not containing x.

Proof If $x \notin J_y = \{a \in S \,/\, a^m \leq y\}$. Then $x^m \nleq y$. Also, if $b \in F \cap J_y$, where, $F = [x) = \{b \in S \,/\, x^m \leq b\}$. is a filter of S, then $x^m \leq b$ and $b^m \leq y$, and hence, by Corollary 7 (iii) of [12], $x^m \leq b^m \leq y$, which implies, $x^m \leq y$, a contradiction. Therefore, $F \cap J_y = \varphi$ Since S is prime, by Theorem 6.3, there exists a prime filter F'such that $F \subseteq F'$ and $F' \cap J_y = \varphi$. Thus, $x \in F$ implies that $x \in F'$, and hence, $x \notin S - F'$. Also, $S - F' \supseteq J_y$ and $S - F'$is a prime ideal of S.

Corollary 6.9. Let S be a prime $(m, m\ +\ 1)$ −semilattice and $y \in S$. Then $J_y = \cap I$, where I is a prime ideal of S such that $I \supseteq J_y$.

Proof Clearly, $J_y \subseteq \cap I$. Suppose, $x \in I$ be such that $x \notin J_y$, then by Theorem 6.8, there exists a prime ideal I' containing J_y and not containing x, which is a contradiction, and hence, $x \in J_y$.

7. EXTENDED IDEAL

In this section, we deal with extended ideals in an $(m, m\ +\ 1)$ −semilattice.

Definition 7.1. Let S be an $(m, m\ +\ 1)$ −semilattice. Let I be a down-set of S and $x \in S$ is an idempotent. An extension of I by x is defined to be the set $< x, I >= \{z \in S/xz \in I\}$.

Remark 7.2. Let S be an $(m, m\ +\ 1)$ −semilattice and $x \in S$ is an idempotent. Let I be a down-set of S. Then $< x, I >$ is also a down-set of S.

Proof Suppose, for $a \in < x, I >$ and $b \in S$ be such that $b \leq a$. Then since x is an idempotent, $xb \leq xa$. Again, since, $xa \in I$, $xb \in I$ and hence $b \in < x, I >$.

Remark 7.3. Let S be an $(m, m\ +\ 1)$ −semilattice. Then, for all $x, y \in S, B_y \subseteq < x^m, B_y >$.

Proof Suppose, $a \in By$, then $a \leq y$ which implies that $x^m a \leq y$. i. e. $x^m a \in B_y$ and hence $a \in < x^m, B_y >$.

Definition 7.4. Let I be an ideal of an $(m, m\ +\ 1)$ −semilattice S and $x \in S$. If < x^m,I > is also an ideal of S, then <x^m, I > is called the extended ideal of I by x.

Proposition 7.5. Let I be a down-set of an $(m, m + 1)$ −semilattice S. If, for all idempotent x in S, $< x, I >$ is an ideal of S, then I must be an ideal of S.

Proof Suppose, for ${x_1}^m, {x_2}^m, \ldots, {x_n}^m \in I$, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$ exists in S. Since, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m$is the least upper bound of $\{{x_1}^m, {x_2}^m, \ldots, {x_n}^m\}$, ${x_i}^m \leq {x_1}^m + {x_2}^m + \cdots + {x_n}^m, \forall\, i = 1,2, \ldots \ldots, n$ and hence, ${x_i}^m({x_1}^m + {x_2}^m + \cdots + {x_n}^m) = {x_i}^m \in I, \forall\, i = 1,2, \ldots \ldots, n$. i. e. ${x_i}^m \in < {x_1}^m + {x_2}^m + \cdots + {x_n}^m, I >, \forall\, i = 1,2, \ldots \ldots, n$, and by hypothesis, $< {x_1}^m + {x_2}^m + \cdots + {x_n}^m, I >$, is an ideal of S. Hence, ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in < {x_1}^m + {x_2}^m + \cdots + {x_n}^m, I >$, which implies that ${x_1}^m + {x_2}^m + \cdots + {x_n}^m \in I$.
Now, we give a new characterization for prime $(m, m\ +\ 1)$ −semilattice in terms of ideals.

Theorem 7.6. For an $(m, m+1)$ −semilattice S, the following are equivalent:
(i) S is a prime $(m, m+1)$ −semilattice.
(ii) $<x^m, I>$ is an ideal of S, for any ideal I of S and $x \in S$.
(iii) $<x^m, I>$ is an ideal of S, for any ideal I of S, such that I is bounded by x.
i. e. $i \leq x$, for any $i \in I$.
(iv) $< x^m, B_y >$ is an ideal, for any $y \leq x$.
(v) $< x^m, B_y >$ is an ideal, for any y, $x \in S$.

Proof (i)⟹(ii)
Suppose, for $x_1{}^m, x_2{}^m, \ldots, x_n{}^m \in < x^m, I >$, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ exists in S. Then $x^m x_i{}^m \in I$, $\forall\, i = 1,2, \ldots \ldots, n$. And, since S is prime, $x^m x_1{}^m + x^m x_2{}^m + \cdots + x^m x_n{}^m$ exists in S. Hence, $x^m x_1{}^m + x^m x_2{}^m + \cdots + x^m x_n{}^m \in I$. i. e. $x^m(x_1{}^m + x_2{}^m + \cdots + x_n{}^m) \in I$ and hence $x_1{}^m + x_2{}^m + \cdots + x_n{}^m \in <x^m, I>$.
(ii)) ⟹ (iii) is obvious.
(iii) ⟹ (iv)
Suppose, $y \leq x$, then $B_y = \{a \in S / a \leq y\}$ is an ideal of S bounded by x. Hence, by (iii), $< x^m, B_y >$ is an ideal of S.
(iv) ⟹ (v)
From (iv), $< x^m, B_y >$is an ideal of S, for any $y \leq x$. By Note 3.8, we have,
$< x^m, B_y > = < x^m, B_{y^m} >$. Also, $< x^m, B_{y^m} > = < x^m, B_{y^m x^m} >$ and $y^m x^m \leq x^m \leq x$. Therefore by (iv), $< x^m, B_{y^m x^m} >$ is an ideal of S and hence, $< x^m, B_y >$is an ideal of S, for any $y, x \in S$.
(v) ⟹ (i)
Suppose, for $x, y, x_1, x_2, \ldots \ldots, x_n \in S$, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$exists in S. Since, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ is the least upper bound of $\{x_1{}^m, x_2{}^m, \ldots, x_n{}^m\}$, $x_i{}^m \leq x_1{}^m + x_2{}^m + \cdots + x_n{}^m, \forall\, i = 1,2, \ldots \ldots, n$. i.e. $x^m x_i{}^m \leq x^m (x_1{}^m + x_2{}^m + \cdots + x_n{}^m)$, $\forall\, i = 1,2, \ldots \ldots, n$.and hence, $x^m(x_1{}^m + x_2{}^m + \cdots + x_n{}^m)$ is an upper bound of $\{x^m x_1{}^m, x^m x_2{}^m, \ldots, x^m x_n{}^m\}$. Suppose, there exists another upper bound y for $\{x^m x_1{}^m, x^m x_2{}^m, \ldots, x^m x_n{}^m\}$, then $x^m x_i{}^m \leq y, \forall\, i = 1,2, \ldots \ldots, n$. i.e. $x^m x_i{}^m \in B_y$, $\forall\, i = 1,2, \ldots \ldots, n$, which implies that $x_i{}^m \in < x^m, B_y >, \forall\, i = 1,2, \ldots \ldots, n$. By (v), $< x^m, B_y >$ is an ideal of S. Therefore, $x_1{}^m + x_2{}^m + \cdots + x_n{}^m < x^m, B_y >$. i. e. $x^m\ (x_1{}^m + x_2{}^m + \cdots + x_n{}^m) \in B_y$, and hence, $x^m\ (x_1{}^m + x_2{}^m + \cdots + x_n{}^m) \leq y$, which shows that $x^m\ (x_1{}^m + x_2{}^m + \cdots + x_n{}^m)$ is the least upper bound of $\{x^m x_1{}^m, x^m x_2{}^m, \ldots, x^m x_n{}^m\}$. i.e. $x^m x_1{}^m + x^m x_2{}^m + \cdots + x^m x_n{}^m$ exists and $x^m x_1{}^m + x^m x_2{}^m + \cdots + x^m x_n{}^m = x^m\ (x_1{}^m + x_2{}^m + \cdots x_n{}^m)$,and hence, S is prime.

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