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Second Hankel Determinant for A New Subclass of The Univalent Analytic Function Defined by A Linear Differential Operator

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ABSTRACT

The main object of this paper is obtained a new sharp upper bound of second Hankel determinant for the subclass of univalent function $S^{m,n,b,¢}(¤)$ in the open unit disk U which is defined by Al-Oboudi Differential Operator.

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INTRODUCTION AND DEFINITIONS

Definition 1.1

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

Which is analytic in the open unit disc $U = \{z \in C : |z| < 1\}$.

The Hankel determinant of f the form (1.1) defined as follows

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & . & . & a_{n+q-1} \\ a_{n+1} & a_{n+2} & . & . & a_{n+q} \\ . & . & . & . & . \\ ' & . & . & . & . \\ a_{n+q-1} & a_{n+q} & . & . & a_{n+2q-2} \end{vmatrix} \quad (1.2)$$

where q & n are fixed positive integers.

The Hankel determinant $H_q(n)$ has been investigated by several authors to study its rate of growth as $n \to \infty$ and to determine the bounds on it for specific values of q & n.

For example, Pommerenke[3] studied Hankel determinant of univalent function satisfy $\left|H_q(n)\right| < Kn^{-(\frac{1}{2}+\ )q+\frac{3}{2}}$.

Later, Hayman[15] studied $\left|H_q(n)\right| < An^{\frac{1}{2}}$, Noor studied Hankel determinant for Bazilevic functions and close to convex function Noor[9], the Fekete and Szego[10] considered the second Hankel determinant $H_2(1) = a_3 - a_2^2$ for univalent functions, Janteng[1] investigated the sharp upper bound for second Hankel determinant $\left|H_2(2)\right| = \left|a_2 a_4 - a_3^2\right|$ for univalent functions and D. Bansal[4] obtained second Hankel determinant for some new class of analytic function.

Recently, Lee *et al.*[12] have studied and obtained bound on $\left|H_2(2)\right| = \left|a_2 a_4 - a_3^2\right|$ for the subclass of Ma-Minda starlike and convex functions. By this brief study, we investigate and obtained sharp upper bound of second Hankel determinant for some new class $S^{m,n,¢,b}(¤)$ of analytic functions.

Definition 1.2

Let $¤(z)$ be an analytic function with positive real part on U with $¤(0) = 1, ¤'(0) > 0$, which maps the unit disc U onto the region Starlike with respect to one which is symmetric with respect to the real axis.

Ma and Minda[16] introduced and studied the class $S^*(\sim)$ consist of function $f(z) \in A$ for which $\frac{zf'(z)}{f(z)} \prec ¤(z), z \in U$. And the class $C(\sim)$ consist of functions $f(z) \in A$ for which $1 + \frac{zf''(z)}{f'(z)} \prec ¤(z), z \in U$.

Ravichandran *et al.*[14] introduced and studied the class $S_b^*(\sim)$ consist of function $f(z) \in A$ for which $1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec ¤(z), z \in U$ and the class $C(\sim)$ consist of functions $f(z) \in A$ for which $1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \prec ¤(z), z \in U$.

Using Alexander[5] transform, it follows that $f(z) \in C(\sim)$ if and only if $zf'(z) \in S^*(\sim)$

Definition 1.3

For $f \in A$, Al-Oboudi[2] introduces the following operator.

$$D^0 f(z) = f(z), \tag{1.3}$$

$$D'f(z) = (1-¢)f(z) + ¢\, zf'(z) = D_¢ f(z), \quad ¢ \geq 0 \tag{1.4}$$

$$D^n f(z) = D_¢(D^{n-1} f(z)), \qquad n \in N = 1,2,3,.... \tag{1.5}$$

$$\therefore D^n f(z) = z + \sum_{j=2}^{\infty} \left[1+(j-1)¢\right]^n a_j z^j, \quad n \in N_0 = N \cup \{0\}. \tag{1.6}$$

If $¢ = 1$, then we get Salagean[7] differential operator.

Definition 1.4

Let $S^{m,n,¢,b}(\sim)$ denoted the sub class of A consisting of functions f which satisfy the inequality

$$\text{Re}\left(1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right) > \sim \tag{1.7}$$

For some $0 \leq \sim < 1$, $b \in C-\{0\}$, $m \in N$, $n \in N_0 = N \cup \{0\}$,and all $z \in U$.

Definition 1.5

Let $S^{m,n,¢,b}(¤)$ denoted the sub class of A consisting of functions f which satisfy the inequality

$$\text{Re}\left(1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right) \prec ¤(z) \tag{1.8}$$

For some $b \in C-\{0\}$, $m \in N$, $n \in N_0 = N \cup \{0\}$,and all $z \in U$.

Definition 1.6

Let two functions f and g are analytic in U, *we* say that $f(z)$ is subordinate to $g(z)$ if there exist a function $w(z)$ analytic in U satisfying $w(0)=0$ and $|w(z)|<1$ such that $f(z) = g(w(z))$ .It is denoted by $f(z) \prec g(z)$, by Schwarz lemma.

2. MAIN RESULT

Lemma 2.1 (see Duren[5])

If $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is a function with positive real part in U ,then sharp estimate, holds.

Lemma 2.2 (see U. Grenander[13])

If $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is a function with positive real part in U , then $2c_2 = c_1^2 + x(4-c_1^2)$ for some $x, |x| \leq 1$, and

$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z$, for some z , $|z| \le 1$.

Theorem 2.3

Let $¤(z) = 1 + B_1 z + B_2 z^2 + ...$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right) \prec ¤(z)$, then $\left|a_2 a_4 - a_3^2\right| \le \frac{b^2 B_1^2}{\left((1+¢)^m - (1+¢)^n\right)^2}$. (2.1)

Then the result is sharp.

Proof

If $f \in S^{m,n,b,¢}(¤)$,then there is a Schwarz function $w(z)$,analytic in U with $w(0)=0$ and $|w(z)| < 1$ in U . such that

$$1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right) = ¤(w(z)) \tag{2.2}$$

Define $P(z)$ by

$$P(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \tag{2.3}$$

Since $w(z)$ is a Schwarz function, it is clear that $\operatorname{Re} P(z) > 0$ and $P(0)=1$.

$$\therefore\ ¤(z) = ¤\left(\frac{P(z)-1}{P(z)+1}\right) = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2$$
$$+\left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right] z^3 + ...$$

Now from (2.2)

$$1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right) = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2$$
$$+\left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right] z^3 + ... \tag{2.4}$$

Equating the like coefficients,

$$\therefore\ a_2 = \frac{bB_1 c_1}{2\left[(1+¢)^m - (1+¢)^n\right]} ,$$

$$a_3 = \frac{bB_1 c_2}{2\left[(1+2¢)^m - (1+2¢)^n\right]}$$
$$+\frac{bB_1 c_1^2}{4\left[(1+2¢)^m - (1+2¢)^n\right]}\left[\frac{((1+¢)^m - (1+¢)^n)b\ B_1}{\left[(1+¢)^m - (1+¢)^n\right]^2} - \left(1 - \frac{B_2}{B_1}\right)\right]$$

and

$$
\begin{aligned}
a_4 = {} & \frac{b^2(B_1B_2-B_1^2)c_1^3\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)}{8\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)} \\
& + \frac{b^3B_1^3c_1^3\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)\left((1+\text{¢})^n\right)}{8\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^2} \\
& - \frac{b^3B_1^3c_1^3\left((1+\text{¢})^{m+2n}-(1+\text{¢})^{3n}\right)}{8\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^3} \\
& + \frac{b(B_3-2B_2+B_1)c_1^3}{8\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)} \\
& + \frac{b^2B_1^2c_1c_2\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)}{4\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)} \\
& + \frac{b(B_2-B_1)c_1c_2}{2\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)} \\
& + \frac{bB_1c_3}{2\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)}
\end{aligned}
$$

$$
a_2a_4-a_3^2=\frac{B_1}{96}\left\{
\begin{aligned}
& \left[\frac{24b^2B_1}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)}\right]c_1c_3 \\
& + \left[
\begin{aligned}
& \frac{12b^3B_1^2\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^2} \\
& + \frac{24b^2(B_2-B_1)}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)} - \frac{24b^2(B_2-B_1)}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2} \\
& - \frac{24b^3B_1^2(1+\text{¢})^n}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2\left((1+\text{¢})^m-(1+\text{¢})^n\right)}
\end{aligned}
\right]c_1^2c_2 \\
& + \left[-\frac{24b^2B_1}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2}\right]c_2^2 \\
& + \left[
\begin{aligned}
& \frac{6b^3(B_1B_2-B_1^2)\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^2} \\
& + \frac{6b^4B_1^3\left((1+2\text{¢})^m(1+\text{¢})^n-(1+2\text{¢})^n(1+\text{¢})^n+(1+\text{¢})^m(1+2\text{¢})^n-(1+\text{¢})^n(1+2\text{¢})^n\right)(1+\text{¢})^n}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^3} \\
& - \frac{6b^4B_1^3\left((1+\text{¢})^{m+2n}-(1+\text{¢})^{3n}\right)}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)^4} + \frac{6b^2(B_3-2B_2+B_1)}{\left((1+3\text{¢})^m-(1+3\text{¢})^n\right)\left((1+\text{¢})^m-(1+\text{¢})^n\right)} \\
& - \frac{6b^2\left(\frac{B_2^2}{B_1}+B_1-2B_2\right)}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2} - \frac{6b^4B_1^3(1+\text{¢})^{2n}}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2\left((1+\text{¢})^m-(1+\text{¢})^n\right)^2} \\
& - \frac{12b^3B_1(B_2-B_1)(1+\text{¢})^n}{\left((1+2\text{¢})^m-(1+2\text{¢})^n\right)^2\left((1+\text{¢})^m-(1+\text{¢})^n\right)}
\end{aligned}
\right]c_1^4
\end{aligned}
\right\} \quad (2.5)
$$

then

$$\left|a_2a_4-a_3^2\right|=T\left|d_1c_1c_3+d_2c_1^2+d_3c_2^2+d_4c_1^4\right| \tag{2.6}$$

where

$$d_1=\left[\frac{24b^2B_1}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}\right],$$

$$d_2=\left[\begin{array}{l}\dfrac{12b^3B_1^2\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^2}\\ +\dfrac{24b^2(B_2-B_1)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}-\dfrac{24b^2(B_2-B_1)}{\left((1+2¢)^m-(1+2¢)^n\right)^2}\\ -\dfrac{24b^3B_1^2(1+¢)^n}{\left((1+2¢)^m-(1+2¢)^n\right)^2\left((1+¢)^m-(1+¢)^n\right)}\end{array}\right],$$

$$d_3=\left[-\frac{24b^2B_1}{\left((1+2¢)^m-(1+2¢)^n\right)^2}\right]$$

and

$$d_4=\left[\begin{array}{l}\dfrac{6b^3(B_1B_2-B_1^2)\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^2}\\ +\dfrac{6b^4B_1^3\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)(1+¢)^n}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^3}\\ -\dfrac{6b^4B_1^3\left((1+¢)^{m+2n}-(1+¢)^{3n}\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^4}+\dfrac{6b^2(B_3-2B_2+B_1)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}\\ -\dfrac{6b^2\left(\frac{B_2^2}{B_1}+B_1-2B_2\right)}{\left((1+2¢)^m-(1+2¢)^n\right)^2}-\dfrac{6b^4B_1^3(1+¢)^{2n}}{\left((1+2¢)^m-(1+2¢)^n\right)^2\left((1+¢)^m-(1+¢)^n\right)^2}\\ -\dfrac{12b^3B_1(B_2-B_1)(1+¢)^n}{\left((1+2¢)^m-(1+2¢)^n\right)^2\left((1+¢)^m-(1+¢)^n\right)}\end{array}\right]$$

and $T=\dfrac{B_1}{96}$.

Since we assume without loss of generality that $c_1>0$ $(c\in[0,2]$ and substituting the value of c_2 & c_3 respectively.

Using lemma (2.3) and (2.6),

$$\left|a_2a_4-a_3^2\right|=\frac{T}{4}\left|\begin{array}{l}c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)\\ +x^2(4-c^2)(-d_1c^2+d_3(4-c^2))+2d_1c(4-c^2)(1-|x|^2)z\end{array}\right| \tag{2.7}$$

Now replace $|x|$ by « and substituting the values d_1, d_2, d_3 & d_4 ,then we get

$$
|a_2a_4 - a_3^2| \le \frac{T}{4}\left|\begin{array}{l}
c^4\left[\begin{array}{l}
\dfrac{24b^3B_1B_2\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^2} \\
-\dfrac{48b^3B_1B_2(1+¢)^n}{\left((1+2¢)^m-(1+2¢)^n\right)^2\left((1+¢)^m-(1+¢)^n\right)}+\dfrac{24b^2B_3}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)} \\
+\dfrac{24b^4B_1^3\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)(1+¢)^n}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^3} \\
+\dfrac{24b^4B_1^3\left((1+¢)^{m+2n}-(1+¢)^{3n}\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^4}-\dfrac{24b^4B_1^3(1+¢)^{2n}}{\left((1+2¢)^m-(1+2¢)^n\right)^2\left((1+¢)^m-(1+¢)^n\right)^2} \\
-\dfrac{24b^2\,\frac{B_2^2}{B_1}}{\left((1+2¢)^m-(1+2¢)^n\right)^2}
\end{array}\right] \\
+2«c^2(4-c^2)\left[\begin{array}{l}
\dfrac{12b^3B_1^2\left((1+2¢)^m(1+¢)^n-(1+2¢)^n(1+¢)^n+(1+¢)^m(1+2¢)^n-(1+¢)^n(1+2¢)^n\right)}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)^2} \\
+\dfrac{24b^2B_2}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}-\dfrac{24b^2B_2}{\left((1+2¢)^m-(1+2¢)^n\right)^2} \\
-\dfrac{24b^3B_1^2(1+¢)^n}{\left((1+2¢)^m-(1+2¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}
\end{array}\right] \\
+«^2(4-c^2)\left[\dfrac{24b^2B_1c^2}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}+\dfrac{24b^2B_1(4-c^2)}{\left((1+2¢)^m-(1+2¢)^n\right)^2}\right] \\
+2c(4-c^2)(1-«^2)\left[\dfrac{24b^2B_1}{\left((1+3¢)^m-(1+3¢)^n\right)\left((1+¢)^m-(1+¢)^n\right)}\right]
\end{array}\right|
$$

$$\equiv F(c, «)$$

If $\dfrac{\partial F}{\partial «} > 0,$ then $F(c, «)$ increasing function of « .

Hence for fixed $c \in [0,2]$ and $« = 1$,the maximum of $F(c, «)$ occurs and

$F(c, «) \equiv F(c,1) \equiv G(c) \quad (say)$

$$\therefore\ |a_2a_4 - a_3^2| \le \frac{b^2B_1^2}{\left((1+2¢)^m-(1+2¢)^n\right)^2} .$$

Letting, $m = 1, n = 0, b = 1$ *and* $¢ = 1$ in theorem (2.3) reduces to See Keong Lee *et al*.[12]

Corollary 2.4

Let $¤(z) = 1 + B_1z + B_2z^2 + ...$, with $B_1 \ne 0$. If $f \in A$ satisfy the inequality $1+\dfrac{1}{b}\left(\dfrac{D^m f(z)}{D^n f(z)}-1\right) \prec ¤(z)$, then $|a_2a_4 - a_3^2| \le \dfrac{B_1^2}{4}$.

Letting, $m = 2, n = 1, b = 1$ *and* $¢ = 1$ in theorem (2.3) reduces to See Keong Lee *et al*.[12]

Corollary 2.5

Let $¤(z) = 1 + B_1 z + B_2 z^2 + ...$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) \prec ¤(z)$, then $\left|a_2 a_4 - a_3^2\right| \leq \frac{B_1^2}{36}$.

Letting, $m = 1, n = 0, b = 1$ *and* $¢ = 1, B_1 = B_2 = B_3 = 2$ in theorem (2.3) reduces to A. Janteng *et al*.[1]

Corollary 2.6

Let $¤(z) = 1 + B_1 z + B_2 z^2 + ...$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) \prec ¤(z)$, then

(i). $\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{16}$. See J. Sokol *et al*.[8].

(ii). $\left|a_2 a_4 - a_3^2\right| \leq \frac{16}{®^4}$. See F. Ronning *et al*.[6].

(iii). $\left|a_2 a_4 - a_3^2\right| \leq {}^{2}$. See R. M, Ali *et al*.[11].

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