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Calderon's Reproducing Formula for Generalized Chebyshev Convolution

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ABSTRACT

Calderon-type reproducing formula for generalized Chebyshev convolution is established using the theory of Chebyshev transform.

Keywords: Calderon formula; Chebyshev convolution; Chebyshev transform.

1. INTRODUCTION

Calderon formula[1] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$f\left(x\right)=\int_0^\infty \left(\varphi_t*\varphi_t*f\right)\left(x\right)\frac{dt}{t}, \tag{1.1}$$

where $\varphi:\mathbb{R}^n\to C \text{ and } \varphi_t(x)=t^{-n}\varphi(x/t),\ t>0.$ For conditions of validity of identity (1.1), we may refer to[1].

Integral transforms involving special functions as kernels have been used by many authors for the construction of wavelets and wavelet transforms. Chebyshev transform is another important transform, which has been used extensively in solving problems of approximation theory by Butzer and Stens[2-4].

First we recall from[3], certain definitions and properties of Chebyshev polynomial and

Chebyshev transforms which form the basis of this work.

Let X denote the space $L_w^p(-1,1)$, $1 \le p < \infty$, or $C(-1,1)$ endowed with the norms

$$\|f\|_p = \|f\|_{L_w^p} = \left[\frac{1}{\pi}\int_{-1}^{1} |f(x)|^p \, w(x)\, dx\right]^{1/p} < \infty, \quad 1 \le p < \infty \tag{1.2}$$

$$\|f\|_C = \sup_{-1\le x\le 1} |f(x)|, \tag{1.3}$$

where

$$w(x) = (1-x^2)^{-1/2}.$$

An inner product on X is given by

$$\langle f, g\rangle_w = \frac{1}{\pi}\int_{-1}^{1} f(x)\overline{g(x)}\, w(x)\, dx \tag{1.4}$$

As usual we denote the Chebyshev polynomial by

$$T_n(x) = \cos(n\cos^{-1}x), \; n \in N_0; x \in (-1,1). \tag{1.5}$$

T_n is a polynomial of degree n, $\|T_n\|_C = 1$ and the T_n satisfy the orthogonality relation

$$\frac{1}{\pi}\int_{-1}^{1} T_n(x)\, T_m(x)\, w(x)\, dx = \begin{cases} 1 & ; \quad m = n = 0 \\ \frac{1}{2} & ; \quad m = n \ne 0 \\ 0 & ; \quad m \ne n \end{cases} \tag{1.6}$$

The Chebyshev transform of a function $f \in X$ is defined by

$$\Im(f)(k) = \hat{f}(k) = \frac{1}{\pi}\int_{-1}^{1} f(x)\, T_k(x)\, w(x)\, dx, \quad k \in N_0 \tag{1.7}$$

The operator $\Im$ associates to each $f \in X$ sequence of real numbers $\left(\hat{f}(k)\right)_{k=0}^{\infty}$, called Fourier – Chebyshev coefficients.

By putting $t = \cos\theta$ in (1.7) and using the fact that

$$\int_{-\pi}^{\pi} f(\cos\theta)\, \sin k\theta \, d\theta = 0,$$

we have

$$\Im(f)(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\cos\theta) e^{-ik\theta} d\theta = F(f o \cos)(k), \tag{1.8}$$

where $F(f o \cos)(k)$ denotes the Fourier transform of the function $f(\cos\theta)$.

The inverse Chebyshev transform is given by

$$\Im\left(f^{\vee}\right)(x) = f(x) = \hat{f}(0) + 2\sum_{k=1}^{\infty}\hat{f}(k)\,T_k(x). \tag{1.9}$$

2. PRELIMINARIES

Lemma 2.1 Assume $f, g \in X$ and $c \in R$, then

(i) $\left|\hat{f}(k)\right| \le \|f\|_1, \quad k \in N_0$;

(ii) $(f+g)^{\wedge}(k) = \hat{f}(k) + \hat{g}(k), \quad (ck)^{\wedge}(k) = c\hat{f}(k)$;

(iii) $\lim_{k\to\infty}\hat{f}(k) = 0$;

(iv) $\hat{f}(k) = 0, k \in N_0$ iff $f(x) = 0$ a.e.;

(v) $$T_n^{\wedge}(k) = \begin{cases} 1 & ; \quad k = n = 0 \\ \frac{1}{2} & ; \quad k = n \ne 0 \\ 0 & ; \quad k \ne n \end{cases}$$

For the proof we may refer to[2].

Let us now define the generalized translation operator τ_y which plays an important role in our investigation. For $f \in X$ and $|y| \le 1$, the translation operator τ_y is defined by

$$\tau_y f(x) = f(x, y) = \frac{1}{2}\left\{ f\left(xy + \left[(1-x^2)(1-y^2)\right]^{1/2}\right) + f\left(xy - \left[(1-x^2)(1-y^2)\right]^{1/2}\right)\right\} \tag{2.1}$$

The translation operator τ_y possesses the following properties.

Lemma 2.2. Let $f \in X$ and $|y| \le 1$, Then

(i) $\|\tau_y f\|_X \le \|f\|_X$;

(ii) $(\tau_y f)(x) = (\tau_x f)(y)$;

(iii) $\lim_{y\to\infty} |\,\|\tau_y f - f\|_X = 0$;

(iv) $\left[\tau_y f\right]^{\wedge}(k) = T_k(y)\,\hat{f}(k)$;

(v) $\tau_y T_n(x) = T_n(y)\,T_n(x), \quad n \in N_0$

The proof can be found in[2].

As in[2], for functions f, g defined and measurable on $(-1,1)$, the generalized Chebyshev convolution is defined by

$$(f*g)(y) = \frac{1}{\pi}\int_{-1}^{1} (\tau_y f)(x)\, g(x)\, w(x)\, \mathrm{dx}. \tag{2.2}$$

Theorem 2.3. If $f \in X, \mathrm{g} \in \mathrm{L}^1_{\mathrm{w}}$, then $(f*g)$ exists (a.e.) and belongs to X. Moreover,

$$\|f*g\|_X \le \|\|f\|_{\mathrm{X}}\|g\|_1; \tag{2.3}$$

$$(f*g)^{\wedge}(k) = \hat{f}(k)\,\hat{\mathrm{g}}(k). \tag{2.4}$$

The proof can be found in[2].

For any $f \in L^2_w(-1,1)$, the following Parseval identity holds for Chebyshev transform:

$$\left|\hat{f}(0)\right|^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left|\hat{f}(k)\right|^2 = \|f\|_2^2 \tag{2.5}$$

In this paper, Chebyshev dilation D_a for a function $\psi \in X$ is defined by

$$D_a\psi(t) = \psi(at), \quad \mathrm{o} < \mathrm{a} \le 1. \tag{2.6}$$

3. CALDERON'S FORMULA

In this section, we obtain Calderon's reproducing identity using the properties of Chebyshev transform and generalized Chebyshev convolution.

Theorem 3.1 Let ϕ and $\psi \in L_w^1(-1,1)$ be such that following admissibility condition holds:

$$\int_{-1}^{1}\hat{\varphi}(\lambda)\,\hat{\psi}(\lambda)\,w(\lambda)\frac{d\lambda}{\lambda} = 1 \tag{3.1}$$

for all $\lambda \in (-1,1)$. Then the following Calderon's reproducing identity holds:

$$f(x) = \int_{-1}^{1}(f*\varphi_a*\psi_a)(x)\,w(a)\frac{da}{a} \qquad \forall f \in L_w^1(-1,1) \tag{3.2}$$

Proof: Taking Chebyshev transform of the right hand side of (3.2), we get

$$\Im\left[\int_{-1}^{1}(f*\varphi_a*\psi_a)(x)\,w(a)\frac{da}{a}\right] = \int_{-1}^{1}\hat{f}(\lambda)\,\hat{\varphi}_a(\lambda)\,\hat{\psi}_a(\lambda)\,w(a)\frac{da}{a}$$

$$= \hat{f}(\lambda)\int_{-1}^{1}\hat{\varphi}_a(\lambda)\,\hat{\psi}_a(\lambda)\,w(a)\frac{da}{a} \tag{3.3}$$

$$= \hat{f}(\lambda) \int_{-1}^{1} \hat{\varphi}_a(a\lambda)\hat{\psi}_a(a\lambda) w(a) \frac{da}{a}$$

$$= \hat{f}(\lambda)$$

Now, by putting $a\lambda = \omega$

$$\int_{-1}^{1} \hat{\varphi}_a(a\lambda)\hat{\psi}_a(a\lambda) w(a) \frac{da}{a} = \int_{-1}^{1} \hat{\varphi}_a(\omega)\hat{\psi}_a(\omega) w(\omega) \frac{d\omega}{\omega} \tag{3.4}$$

$$=1.$$

Hence the result follows.

Theorem 3.2 Suppose $\varphi \in L_w^1(-1,1)$ is real valued and satisfies

$$\int_{-1}^{1} \left[\hat{\varphi}_a(a\lambda)\right]^2 w(a) \frac{da}{a} = 1 \tag{3.5}$$

For $f \in L_w^1(-1,1) \cap L_w^2(-1,1)$ suppose that

$$f_{\varepsilon,\delta}(x) = \int_{-1}^{1} (f * \varphi_a * \varphi_a)(x) w(a) \frac{da}{a} \tag{3.6}$$

Then $\| f - f_{\varepsilon,\delta} \|_2 \to 0$ as $\varepsilon \to -1$ and $\delta \to 1$

Proof: Taking Chebyshev transform of both sides of (3.6) and using Fubini's theorem, we get

$$\hat{f}_{\varepsilon\delta}(\lambda) = \hat{f}(\lambda) \int_{-1}^{1} \left[\hat{\varphi}_a(a\lambda)\right]^2 w(a) \frac{da}{a} \tag{3.7}$$

By[1], we have

$$\|\varphi_a * \varphi_a * f\|_2 \le \|\varphi_a * \varphi_a\|_1 \|f\|_2$$

$$\le \|\varphi_a\|_1^2 \|f\|_2 . \tag{3.8}$$

Now using above inequality and Minkowski's inequality [5, page 41],we get

$$\|f\|_2^2 = \int_{-1}^{1} dx \left| \int_{\varepsilon}^{\delta} (\varphi_a * \varphi_a * f)(x) w(a) \frac{da}{a} \right|^2$$

$$\le \int_{\varepsilon}^{\delta} \int_{-1}^{1} \left| (\varphi_a * \varphi_a * f)(x) \right|^2 dx w(a) \frac{da}{a}$$

$$\le \int_{\varepsilon}^{\delta} \left\| (\varphi_a * \varphi_a * f)(x) \right\|_2 w(a) \frac{da}{a} \tag{3.9}$$

$$\le \|\varphi\|_1^2 \|f\|_2 \int_\varepsilon^\delta \frac{dt}{t}$$

$$= \|\varphi_a\|_1^2 \|f\|_2 \log\left(\frac{\delta}{\varepsilon}\right).$$

Hence by Parseval formula, we get

$$\lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} \left\| f - f_{\varepsilon,\delta} \right\|_2^2 = \lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} \left\| \hat{f} - \hat{f}_{\varepsilon,\delta} \right\|_2^2$$

$$= \lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} \int_{-1}^{1} \left| \hat{f}(\lambda) \left(1 - \int_\varepsilon^\delta \left[\hat{\phi}(a\lambda)\right]^2 w(a) \frac{da}{a}\right)\right|^2 dx \tag{3.10}$$

$$= 0$$

Since $\left| \hat{f}(\lambda) \left(1 - \int_\varepsilon^\delta \left[\hat{\varphi}(a\lambda)\right]^2 w(a) \frac{da}{a}\right)\right| \le \left|\hat{f}(\lambda)\right|$, therefore by the dominated convergence theorem, the result follows.

The reproducing identity (3.2) holds in the point wise sense under different set of nice conditions.

Theorem 3.3 Suppose $f, \hat{f} \in L_w^1(-1,1)$. Let $\varphi \in L_w^1(-1,1)$ be real valued and satisfies

$$\int_{-1}^{1} \left[\hat{\varphi}(a\lambda)\right]^2 w(a) \frac{da}{a} = 1, \quad \lambda \in R - \{0\}. \tag{3.11}$$

Then

$$\lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} \int_\varepsilon^\delta (f * \varphi_a * \varphi_a)(x)\, w(a) \frac{da}{a} = f(x). \tag{3.12}$$

Proof: Let

$$f_{\varepsilon,\delta}(x) = \int_\varepsilon^\delta (f * \varphi_a * \varphi_a)(x)\, w(a) \frac{da}{a} \tag{3.13}$$

By[1], we have

$$\|\varphi_a * \varphi_a * f\|_1 \le \|\varphi_a * \varphi_a\|_1 \|f\|_1$$

$$\le \|\varphi_a\|_1^2 \|f\|_1 \tag{3.14}$$

Now,

$$\left\| f_{\varepsilon,\delta} \right\|_1 = \int_{-1}^{1} dx \left| \int_\varepsilon^\delta (\varphi_a * \varphi_a * f)(x)\, w(a) \frac{da}{a} \right|$$

$$\le \int_{\varepsilon}^{\delta}\int_{-1}^{1}\left|(\varphi_a*\varphi_a*f)(x)\right| dx\, w(a)\frac{da}{a}$$

$$\le \int_{\varepsilon}^{\delta}\left\|(\varphi_a*\varphi_a*f)(x)\right\|_1 w(a)\frac{da}{a} \tag{3.15}$$

$$\le \|\varphi_a\|_1^2\,\|f\|_1\int_{\varepsilon}^{\delta}\frac{dt}{t}$$

$$= \|\phi_a\|_1^2\,\|f\|_1 \log\left(\frac{\delta}{\varepsilon}\right).$$

Therefore, $f_{\varepsilon,\delta}\in L_w^1(-1,1)$. Also using Fubini's, we get theorem and taking Chebyshev transform of (3.13), we get

$$\hat{f}_{\varepsilon,\delta}(\lambda) = \int_{-1}^{1} k_\lambda(x\lambda)\left(\int_{\varepsilon}^{\delta}(\varphi_a*\varphi_a*f)(x)\, w(a)\frac{da}{a}\right)dx$$

$$= \int_{\varepsilon}^{\delta}\int_{-1}^{1} k_\lambda(x\lambda)(\varphi_a*\varphi_a*f)(x)\, dx\, w(a)\frac{da}{a} \tag{3.16}$$

$$= \int_{\varepsilon}^{\delta}\hat{\varphi}_a(\lambda)\hat{\varphi}_a(\lambda)\hat{f}(\lambda)\, w(a)\frac{da}{a}$$

$$= \hat{f}(\lambda)\int_{\varepsilon}^{\delta}\left[\hat{\varphi}(a\lambda)\right]^2 w(a)\frac{da}{a}.$$

Therefore, by (3.11), $\left|\hat{f}_{\varepsilon,\delta}(\lambda)\right| \le \left|\hat{f}(\lambda)\right|$.

It follows that $\hat{f}_{\varepsilon,\delta}\in L_w^1(-1,1)$.By inversion, we have

$$f(x) - f_{\varepsilon,\delta}(x) = \int_{-1}^{1} k_\lambda(x\lambda)\left[\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda)\right] w(\lambda)\, d\lambda,\; x\in(-1,1). \tag{3.17}$$

Putting

$$h_{\varepsilon,\delta}(\lambda : x) = k_\lambda(x\lambda)\left[\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda)\right]$$

$$= \hat{f}(\lambda)\, k_\lambda(x\lambda)\left[1 - \int_{\varepsilon}^{\delta}\left[\hat{\varphi}(a\lambda)\right]^2 w(a)\frac{da}{a}\right] \tag{3.18}$$

we get

$$f(x) - f_{\varepsilon,\delta}(x) = \int_{-1}^{1} k_\lambda(x\lambda)\left[\hat{f}(\lambda) - \hat{f}_{\varepsilon,\delta}(\lambda)\right] w(\lambda)\, d\lambda \tag{3.19}$$

$$= \int_{-1}^{1} h_{\varepsilon,\delta}\ \lambda : x\ \ w\ \lambda\ \ d\lambda .$$

Now using (3.11) in (3.18), we get

$$\lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} h_{\varepsilon,\delta}\ \ \lambda : x\ \ = 0,\ \ \lambda \in R - \ 0\ \ . \tag{3.20}$$

Since $\left| h_{\varepsilon,\delta}\ \ \lambda : x\ \right| \le \left| \hat{f}\ \ \lambda\ \right|$, the Lebsegue dominated convergence theorem yields

$$\lim_{\substack{\varepsilon\to -1\\ \delta\to 1}} \left[f(x) - f_{\varepsilon,\delta}\ \ x\ \right] = 0,\ \ \forall\ \ x \tag{3.21}$$

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