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Strongly Multiplicative Labeling of Double Circus and Mongolian Networks

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ABSTRACT

A graph $G = (V(G), E(G))$ with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, \dots, p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct[5].

In this paper we prove that the Double circus and Mongolian network are strongly multiplicative for all $n \geq 2$.

Keywords: Graph labeling, strongly multiplicative labeling, Double circus and Mongolian networks.

1. INTRODUCTION

Graph labeling concerns the assigning of values, usually represented by integers, to the edges and /or vertices of a graph[1]. It plays an important role in Neural Networks, Communication Networks, Circuit Analysis, Coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties and also used in the study of X-ray Crystallography, etc.[2]. Graph labeling serves as a frontier between number theory and structure of graphs[4].

In this paper we prove that double circus and Mongolian networks are strongly multiplicative.

2. STRONGLY MULTIPLICATIVE LABELING OF DOUBLE CIRCUS NETWORK

Definition 2.1: The double circus network $DCT(n)$ is defined as a graph obtained from the two copies of ladder graphs $P_n \times P_2$ by adding a new vertex u between the grids and joining every vertex of the top row of one grid and the bottom row of the other grid to the centre vertex u.

Remark 2.1: (i) The number of vertices in double circus network $DCT(n)$ is $4n + 1$.
(ii) The number of edges in double circus network $DCT(n)$ is $4(2n - 1)$.

Theorem 2.1 The double circus tent $DCT(n)$ is strongly multiplicative for all $n \geq 2$.
Proof:
To prove that the double Circus network $DCT(n)$ is strongly multiplicative.
Define the vertex set $V = \{v_i / 1 \leq i \leq 4n\} \cup \{u\}$ and the edge set as
$E = E_1 \cup E_2 \cup E_3$ where
$E_1 = \{e_i = (v_i, v_{i+1}) / 1 \leq i \leq n-1, n + 1 \leq i \leq 2n-1, 2n + 1 \leq i \leq 3n-1,$
$3n + 1 \leq i \leq 4n-1\}$
$E_2 = \{e_i = (v_i, v_{n+i}) / 1 \leq i \leq n, 2n + 1 \leq i \leq 3n\}$
$E_3 = \{e_i = (v_i, u) / n + 1 \leq i \leq 2n, 2n+1 \leq i \leq 3n\}$.
Define the vertex labeling of $DCT(n)$ as follows for all $1 \leq i \leq n$
$f(v_i) = 2i - 1$; $f(v_{n+i}) = 2i$; $f(v_{2n+i}) = 2(n + i)$;
$f(v_{3n+i}) = 2(n + i) - 1$; $f(u) = 4n + 1$

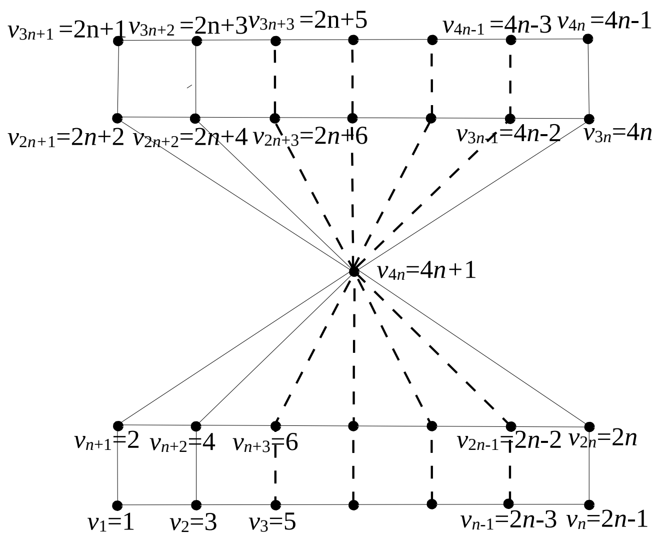

Figure 1: Double Circus Network $DCT(n)$

We shall prove that all the edge labeling in E are distinct.

Define an edge induced function $g: E_1 \to N$ such that for all $e_i \in E_1$
$g(e_i) = f(v_i)\, f(v_{i+1})$, $1 \leq i \leq 4n$
We shall show that the labeling of edges within E_1 is distinct :

If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.
Case 1: For $i \neq p, 1 \leq i, p \leq n-1$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$
$(2i - 1)(2(i + 1) - 1) = (2p - 1)(2(p + 1) - 1)$
$4i^2 - 1 = 4p^2 - 1 \Rightarrow i = p$, a contradiction for i.
Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n - 1$
Case 2: For $i \neq p,\ n + 1 \leq i, p \leq 2n - 1$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$
$(2i)(2(i + 1)) = (2p)(2(p + 1))$
$i^2 + i = p^2 + p \Rightarrow i = -1 - p$, a contradiction for i.
Hence $g(e_i) \neq g(e_p), \forall\, n + 1 \leq i, p \leq 2n - 1$.
similarly edge labeling within E_1 can be proved to be distinct for
$2n + 1 \leq i, p \leq 3n - 1$ and $3n + 1 \leq i, p \leq 4n - 1$.
Case 3: For $i \neq p, 1 \leq i \leq n - 1$ and $2n + 1 \leq p \leq 3n - 1$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$
$(2i - 1)(2(i + 1) - 1) = 2(n + p)[2(n + p + 1)]$
$4i^2 - 1 = x(x + 2)$, *where* $x = 2(n + p)$
$i = \frac{x+1}{2} = n + p + \frac{1}{2}$, a contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for $1 \leq i \leq n - 1$ and $2n + 1 \leq p \leq 3n - 1$.

Case 4: For $i \neq p, n + 1 \leq i \leq 2n - 1$ *and* $3n + 1 \leq p \leq 4n - 1$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$
$(2i)(2(i + 1)) = [2(n + p) - 1][2(n + p) + 1]$
$4i(i + 1) = [4(n + p)^2 - 1]$
$\Rightarrow i = n + p - \frac{1}{2}$, contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for $n + 1 \leq i \leq 2n - 1$ and $3n + 1 \leq p \leq 4n - 1$.
similarly other cases can be proved.
Hence all edge labelings in E_1 are distinct.
We shall show that the labeling of edges within E_2 *is distinct* :

Define an edge induced function $g : E_2 \to N$ such that for all $e_i \in E_2$,
$g(e_i) = f(v_i)f(v_{n+i}), 1 \leq i \leq n, 2n + 1 \leq i \leq 3n.$
If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$.
Case 1: For $i \neq p, 1 \leq i, p \leq n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{n+i}) = g(v_p, v_{n+p})$
$f(v_i)f(v_{n+i}) = f(v_p)f(v_{n+p})$
$(2i - 1)2i = (2p - 1)2p$
$2i^2 - i = 2p^2 - p \Rightarrow i = \frac{1}{2} - p$, a contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i, p \leq n$.
Similarly we can prove $g(e_i) \neq g(e_p)$ for $2n + 1 \leq i, p \leq 3n$.
Case 2: For $i \neq p, 1 \leq i \leq n$ and $2n + 1 \leq p \leq 3n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{n+i}) = g(v_p, v_{n+p})$
$f(v_i)f(v_{n+i}) = f(v_p)f(v_{n+p})$
$(2i - 1)2i = 2(n + p)[2(n + p) - 1]$
$x(x - 1) = y(y - 1), where\ x = 2i, y = 2(n + p)$
$i = \frac{1}{2} - (n + p)$, a contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq n$ and $2n + 1 \leq p \leq 3n$.
Hence all the labeling of edges in E_2 is distinct.
We shall show that the labeling of edges within E_3 is distinct:
Define an edge induced function $g : E_3 \to N$ such that for all $e_i \in E_3$
$g(e_i) = f(v_i)f(u), n + 1 \leq i \leq 2n$ and $2n + 1 \leq i \leq 3n$
If e_i and e_p are distinct edges in E_3 then to prove $g(e_i) \neq g(e_p)$.

Case 1: For $i \neq p, n + 1 \leq i, p \leq 2n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, u) = g(v_p, u)$
$f(v_i)f(u) = f(v_p)f(u)$
$2i(4n + 1) = 2p(4n + 1) \Rightarrow i = p,$ contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for all $n + 1 \leq i, p \leq 2n$.
Case 2: For $i \neq p, n + 1 \leq i \leq 2n$ and $2n + 1 \leq p \leq 3n$.
Assume that $g(e_i) = g(e_p)$
$g(v_i, u) = g(v_p, u)$
$f(v_i)f(u) = f(v_p)f(u)$
$2i(4n + 1) = 2(n + p)(4n + 1) \Rightarrow p = i - n$, contradiction for p.

Hence $g(e_i) \neq g(e_p)$, for all $n+1 \leq i \leq 2n$ and $2n+1 \leq p \leq 3n$.
Hence all the edge labeling set in E_3 is distinct.
We shall show that the labeling of edges E_1 and E_2 are distinct:
If e_i and e_p are distinct edges in E_1 and E_2 to prove that $g(e_i) \neq g(e_p)$.
Case 1: For $i \neq p, 1 \leq i \leq n-1$ and $1 \leq p \leq n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{n+p})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{n+p})$
$(2i-1)(2(i+1)-1) = (2p-1)2p$
$4i^2 - 1 = 4p^2 - 2p \Rightarrow i = \sqrt{(p-\frac{1}{4})^2 + \frac{3}{16}}$, a contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq n-1$ and $1 \leq p \leq n$.
Case 2: For $i \neq p, n+1 \leq i \leq 2n-1$ and $1 \leq p \leq n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{n+p})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{n+p})$
$2i[2(i+1)] = (2p-1)2p$

$$\left(i+\frac{1}{2}\right)^2 = \left(p-\frac{1}{4}\right)^2 + \frac{3}{16} \Rightarrow i = \sqrt{\left(p-\frac{1}{4}\right)^2 + \frac{3}{16}} - \frac{1}{2}$$

This is a contradiction for i.
Hence $(e_i) \neq g(e_p)$, for all $n+1 \leq i \leq 2n-1$ and $1 \leq p \leq n$.
Case 3: For $i \neq p$, $1 \leq i \leq n-1$ and $3n+1 \leq p \leq 4n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{i+1}) = g(v_p, v_{n+p})$
$f(v_i)f(v_{i+1}) = f(v_p)f(v_{n+p})$
$2i - 1(2(i+1)-1) = (2(n+p)-1)2(n+p)$

$$i = \frac{\sqrt{\left[2(n+p)-\frac{1}{2}\right]^2 + \frac{3}{4}}}{2}$$

This is a contradiction for i.
Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq n-1$ and $3n+1 \leq p \leq 4n$.
For other cases similar proof follows.
Hence all the edge labeling set in E_1 and E_2 are distinct.
We shall show that the labeling of edges E_2 and E_3 are distinct:
If e_i and e_p are distinct edges in E_2 and E_3 to prove that $g(e_i) \neq g(e_p)$.
Case 1: For $i \neq p, 1 \leq i \leq n$ and $n+1 \leq p \leq 2n$
Assume that $g(e_i) = g(e_p)$

$g(v_i, v_{n+i}) = g(v_p, u)$
$f(v_i)f(v_{n+i}) = f(v_p)f(u)$
$2i(2i - 1) = 2p(4n + 1)$
$\Rightarrow p = \frac{2i^2 - i}{4n+1}$, contradiction for p.
Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq n$ and $n + 1 \leq p \leq 2n$.
Case 2: For $i \neq p, 2n + 1 \leq i, p \leq 3n$
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{n+i}) = g(v_p, u)$
$f(v_i)f(v_{n+i}) = f(v_p)f(u)$
$2(n + i)[2(n + i) - 1] = 2(n + p)(4n + 1)$
$\Rightarrow p = \frac{(n+i)[2(n+i)-1]}{(4n+1)} - n$, a contradiction for p.
Hence $g(e_i) \neq g(e_p)$, for all $2n + 1 \leq i, p \leq 3n$.
Case 3: For $i \neq p, 2n + 1 \leq i \leq 3n \, and \, n + 1 \leq p \leq 2n$.
Assume that $g(e_i) = g(e_p)$
$g(v_i, v_{n+i}) = g(v_p, u)$
$f(v_i)f(v_{n+i}) = f(v_p)f(u)$
$2(n + i)[2(n + i) - 1] = 2p(4n + 1)$
$\Rightarrow p = \frac{(n+i)[2(n+i)-1]}{(4n+1)}$, contradiction for p.
Hence $g(e_i) \neq g(e_p)$, for all $2n + 1 \leq i \leq 3n$ and $n + 1 \leq p \leq 2n$.
Thus the induced edge labeling in E_2 and E_3 is distinct. Hence all the induced edge labeling in E are distinct. Therefore double circus network $DCT(n)$ is strongly multiplicative for all $n \geq 2$.
□

3. STRONGLY MULTIPLICATIVE LABELING OF MONGOLIAN NETWORK

Definition 3.1:
A Mongolian graph for any m and n denoted $M_{m,n}$ is defined as the graph obtained from $P_m \times P_n$ by adding a new vertex u above the grid and joining every vertex of the top row to the new vertex u [6]
Remark 3.1:
(i) The number of vertices in Mongolian graph $M_{m,n}$ is $mn + 1$.
(ii) The number of edges in Mongolian graph $M_{m,n}$ is $m(n - 1) + m(n + 1)$.[3]

Theorem 3.1:
The Mongolian Network $M_{m,n}$ is strongly multiplicative for any n and for all m even.
Proof:
Let $M_{m,n}$ be a Mongolian network with vertex set
$V = \{v_1, v_2 \ldots v_n, v_{n+1}, v_{n+2} \ldots v_{2n}, v_{2n+1} \ldots v_{3n} \ldots v_{(m-1)n+1}, \ldots v_{mn}\} \cup \{u = v_{mn+1}\}$

and the edge set $E = E_1 \cup E_2 \cup E_3$ where
$E_1 = \{e_i = (v_{(j-1)n+i}, v_{(j-1)n+i+1}) \ / \ 1 \le i \le n - 1, 1 \le j \le m\}$,
$E_2 = \{e_i = (v_{(j-1)n+i}, v_{jn+i}) \ / \ 1 \le i \le n, 1 \le j \le m - 1\}$,
$E_3 = \{e_i = (v_{(m-1)n+i}, v_{mn+1}) \ / \ 1 \le i \le n\}$.
To prove that $M_{m,n}$ is strongly multiplicative,
Define a bijective mapping $f : V \to \{1, 2, 3, \dots, mn + 1\}$ as follows:

$$f(v_{(j-1)n+i}) = \begin{cases} 2i - 1 + n(j-1) & j - odd \\ 2i + (j-2)n & j - even \end{cases} \quad 1 \le i \le n, 1 \le j \le m \ \&$$

$$f(u) = mn + 1$$

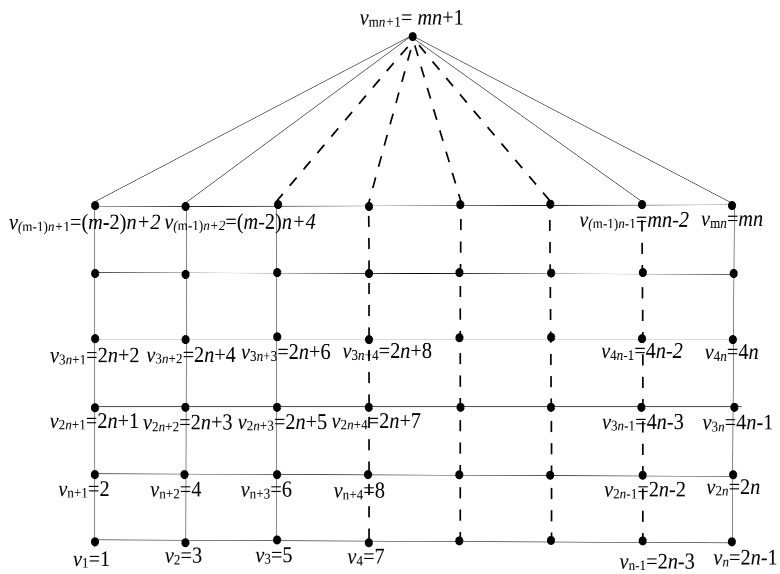

Figure 2: Mongolian Network $M_{m,n}$ when m is even

We shall first show that the labeling in edge set E_1 is distinct:
Define an edge induced function $g: E_1 \to N$ such that $1 \le i \le n - 1$
$g(e_i) = f(v_{(j-1)n+i}) f(v_{(j-1)n+i+1})$ for all $e_i \in E_1$ and $(v_{(j-1)n+i}, v_{(j-1)n+i+1}) \in$ V.
If e_i and e_p are distinct edges in E_1 then to prove that $g(e_i) \neq g(e_p)$.
Case 1: For $i < p, 1 \le i, p \le n - 1, 1 \le j, q \le m$ and where j, q are both odd.
Assume that $g(e_i) = g(e_p)$
$g(v_{(j-1)n+i}, v_{(j-1)n+i+1}) = g(v_{(q-1)n+p}, v_{(q-1)n+p+1})$
$f(v_{(j-1)n+i}) f(v_{(j-1)n+i+1}) = f(v_{(q-1)n+p}) f(v_{(q-1)n+p+1})$
$[2i - 1 + (j-1)n][2(i+1) - 1 + (j-1)n]$
$= [2p - 1 + (q-1)n][2(p+1) - 1 + (q-1)n]$
$(x-1)(x+1) = (y-1)(y+1)$ where $x = [2i + (j-1)n]$ & $y = [2p + (q-1)n]$
$\Rightarrow x = y, 2i + (j-1)n = 2p + (q-1)n$

$\Rightarrow i = p + \frac{n(q-j)}{2}$, contradiction for i.
Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n - 1$.
Case 2: For $i < p, 1 \leq i, p \leq n - 1, 1 \leq j, q \leq m$ and where j, q are both even.
Assume that $g(e_i) = g(e_p)$
$g\big(v_{(j-1)n+i}, v_{(j-1)n+i+1}\big) = g(v_{(q-1)n+p}, v_{(q-1)n+p+1})$
$f\big(v_{(j-1)n+i}\big)f\big(v_{(j-1)n+i+1}\big) = f(v_{(q-1)n+p})f(v_{(q-1)n+p+1})$
$[2i + (j - 2)n][2(i + 1) + (j - 2)n]$
$= [2p + (q - 2)n][2(p + 1) + (q - 2)n]$
$x(x + 2) = y(y + 2) \Rightarrow x + y = -2$, a contradiction for i.
Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n - 1$.
Case 3: For $i < p, 1 \leq i, p \leq n - 1, 1 \leq j, q \leq m$ and j is odd and p is even.
Assume that $g(e_i) = g(e_p)$
$g\big(v_{(j-1)n+i}, v_{(j-1)n+i+1}\big) = g(v_{(q-1)n+p}, v_{(q-1)n+p+1})$
$f\big(v_{(j-1)n+i}\big)f\big(v_{(j-1)n+i+1}\big) = f(v_{(q-1)n+p})f(v_{(q-1)n+p+1})$
$[2i - 1 + (j - 1)n][2(i + 1) - 1 + (j - 1)n] = [2p + (q - 2)n][2(p + 1) + (q - 2)n]$
$(x - 1)(x + 1) = y(y + 2) \Rightarrow x = y + 1$
$\Rightarrow i = p + \frac{n[q-j-1]+1}{2}$, a contradiction for i.
Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i, p \leq n - 1$.
Hence all the induced edge labeling in E_1 are distinct.
We shall show that the labeling in edge set E_2 are distinct:
Define an edge induced function $g : E_2 \rightarrow N$ such that
$g(e_i) = f(v_{(j-1)n+i})f(v_{jn+i})_i)$ for all $e_i \in E_2$ and $(v_{(j-1)n+i}, v_{jn+i}) \in V$,$1 \leq i \leq n$.
If e_i and e_p are distinct edges in E_2 then to prove that $g(e_i) \neq g(e_p)$.
For $i < p, 1 \leq i, p \leq n, 1 \leq j, q \leq m - 2$ and where j is odd and q is even.
Assume that $g(e_i) = g(e_p)$
$g\big(v_{(j-1)n+i}, v_{jn+i}\big) = g(v_{(q-1)n+p}, v_{qn+p})$
$f\big(v_{(j-1)n+i}\big)f\big(v_{jn+i}\big) = f(v_{(q-1)n+p})f(v_{qn+p})$
$[2i - 1 + (j - 1)n][2i + (j - 2)n] = [2p + (q - 2)n][2p - 1 + (q - 1)n]$
$(x - n - 1)(x - 2n) = (y - 2n)(y - n - 1) \Rightarrow x = 1 + 3n - y$
$i = \frac{1+n(3-q-j)}{2} - p$, a contradiction for i.

Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n, 1 \leq j, q \leq m - 2$.
Proof holds similarly when q is odd and j is even

Hence all the induced edge labeling in E_2 are distinct.
We shall show that the labeling in edge set E_3 is distinct.
Define an edge induced function $g : E_3 \rightarrow N$ such that for all $1 \leq i \leq n$
$g(e_i) = f(v_{(m-1)n+i})f(v_{mn+1})$ for all $e_i \in E_3$ and $(v_{(m-1)n+i}, v_{mn+1}) \in V$.

If e_i and e_p are distinct edges in E_3 then to prove $g(e_i) \neq g(e_p)$.
For $i < p, 1 \leq i, p \leq n, j, q = m - 1$ and where j and q both even.
Assume that $g(e_i) = g(e_p)$

$g(v_{(m-1)n+i}, v_{mn+1}) = g(v_{(m-1)n+p}, v_{mn+1})$

$f(v_{(m-1)n+i}) f(v_{mn+1}) = f(v_{(m-1)n+p}) f(v_{mn+1})$

$[2i + (m - 2)n][mn + 1] = [2p + (m - 2)n][mn + 1]$

$\Rightarrow i = p$, a contradiction for i.
Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n, j, q = m - 1$.
Hence all the induced edge labeling in E_3 are distinct.

We shall show that the labeling in edge set E_2 and E_3 are distinct:

If $e_i \in E_2$ and $e_p \in E_3$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
For $i < p, 1 \leq i, p \leq n,\ q = m - 1$ and where j is even.
Assume that $g(e_i) = g(e_p)$

$g(v_{(j-1)n+i}, v_{jn+i}) = g(v_{(m-1)n+p}, v_{mn+1})$

$f(v_{(j-1)n+i}) f(v_{nj+i}) = f(v_{(m-1)n+p}) f(v_{mn+1})$

$[2i - 1 + (j - 1)n][2i + (j - 2)n] = [2p + (m - 2)n][mn + 1]$

$(x - 2n)(x - 1 - n) = [2p + (m - 2)n][mn + 1]$ $where\ x = 2i + jn$

$\Rightarrow y = 2n + \frac{(x-2n)(x-1-n)}{mn+1}$

$p = -\frac{nm}{2} + n + \frac{(2i+jn-1-n)(2i+jn-2n)}{2(nm+1)}$, a contradiction for p.

Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n - 1$.
Hence all the induced edge labeling in E_2 and E_3 are distinct.

We shall show that the labeling in edge set E_1 and E_2 are distinct:

If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
Case 1: For $i < p, 1 \leq p \leq n, 1 \leq i \leq n - 1$ and where j, q are both odd.
Assume that $g(e_i) = g(e_p)$

$g(v_{(j-1)n+i}, v_{(j-1)n+i+1}) = g(v_{(q-1)n+p}, v_{qn+p})$

$f(v_{(j-1)n+i}) f(v_{(j-1)n+i+1}) = f(v_{(q-1)n+p}) f(v_{qn+p})$

$[2i - 1 + (j - 1)n][2(i + 1) - 1 + (j - 1)n] = [2p + (q - 2)n][2p - 1 + (q - 1)n]$

$2i = \sqrt{[2p + (q - 2)n][2p - 1 + (q - 1)n] + 1} - n(j - 1)$, a contradiction for i.

Hence $g(e_i) \neq g(e_p). \forall\ 1 \leq i, p \leq n - 1$.

Case 2: For $i < p, 1 \leq p \leq n, 1 \leq i \leq n - 1, 1 \leq j, q \leq m - 1$ and where j, q are both even. Proof holds similarly as in case 1.

Hence all the induced edge labeling in E_1 and E_2 are distinct. Thus the labeling of edges of E_1, E_2 and that of E_3 are all distinct. Also the induced edge labeling of E_1 & E_2 and E_2 & E_3 are distinct.

Hence the Mongolian Network has strongly multiplicative labeling for all n and for all even m. □

Theorem 3.2:
The Mongolian Network $M_{m,n}$ is strongly multiplicative for any n and m- odd.
Proof.
Let $M_{m,n}$ be a Mongolian network with the vertex set
V = {v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, ..., v_{2n}, v_{2n+1}, ..., v_{3n}, ... $v_{(m-1)n+1}$, ... v_{mn}} ∪ {u = v_{mn+1}}
and the edge set $E = E_1 \cup E_2 \cup E_3$ where
$E_1 = \{e_i = (v_{(j-1)n+i}, v_{(j-1)n+i+1}) \ / \ 1 \leq i \leq n - 1, 1 \leq j \leq m\}$,
$E_2 = \{e_i = \left(v_{(j-1)n+i}, v_{jn+i}\right) \ / \ 1 \leq i \leq n, 1 \leq j \leq m-1\}$,
$E_3 = \{e_i = (v_{(m-1)n+i}, v_{mn+1}) \ / \ 1 \leq i \leq n\}$.
To prove that $M_{m,n}$, is strongly multiplicative,
Define a bijective mapping $f : V \rightarrow \{1, 2, 3, \ldots, mn + 1\}$ as follows:

$$f\left(v_{(j-1)n+i}\right) = \begin{cases} 2i - 1 + n(j-1) & j - odd \\ 2i + (j-2)n & j - even \end{cases} \quad 1 \leq i \leq n, 1 \leq j \leq m-1,$$

$$f\left(v_{(m-1)n+i}\right) = (m-1)n + i \qquad \& \qquad f(u) = mn + 1$$

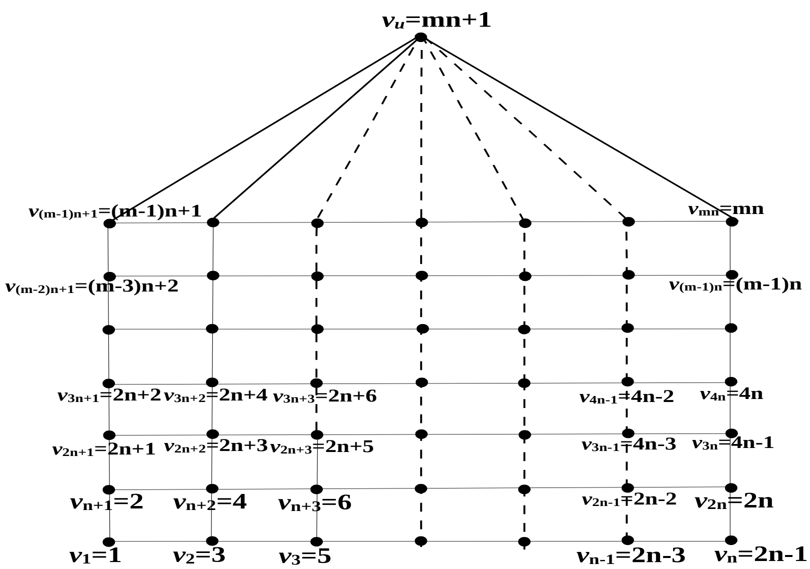

Figure 3: Mongolian Tent $M_{m,n}$, m − odd

We shall first show that the labeling in edge set E_1 is distinct.
Define an edge induced function $g : E_1 \rightarrow N$ such that $1 \leq i \leq n - 1$.
$g(e_i) = f(v_{(j-1)n+i})f(v_{(j-1)n+i+1})$ for all $e_i \in E_1$ and $(v_{(j-1)n+i}, v_{(j-1)n+i+1}) \in$ V.
If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.
Case 1, 2, 3 follows as in the above theorem when m- even
Case 4: For $i < p, 1 \leq i, p \leq n - 1, 1 \leq j \leq m, q = m - 1$ and where *j* is odd.
Assume that $g(e_i) = g(e_p)$

$g\big(v_{(j-1)n+i}, v_{(j-1)n+i+1}\big) = g(v_{(m-1)n+p}, v_{(m-1)n+p+1})$
$f\big(v_{(j-1)n+i}\big)f\big(v_{(j-1)n+i+1}\big) = f\big(v_{(m-1)n+p}\big)f(v_{(m-1)n+p+1})$
$[2i-1+(j-1)n][2(i+1)-1+(j-1)n] = [(m-1)n+p][(m-1)n+p+1]$
$(x-1)(x+1) = y(y+1)$ where $x = [2i+(j-1)n]$ and $y = [(m-1)n+p]$
$x = \sqrt{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}, \Rightarrow 2i = -(j-1)n + \sqrt{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}$ a contradiction for i.
Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i,p \leq n, 1 \leq j \leq m, q = m-1$.
When j is even, similarly we can prove that $g(e_i) \neq g(e_p)$, for all $e_i \in E_1$.
Hence all the induced edge labeling in E_1 are distinct.
We shall show that the labeling in edge set E_2 are distinct:
Define an edge induced function $g : E_2 \to N$ such that
$g(e_i) = f\big(v_{(m-2)n+i}\big)f\big(v_{(m-1)n+i}\big)$ for all $e_i \in E_2$.
If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$.
For $i < p, 1 \leq i,p \leq n, j = m-2, q = m-1$ and where j is even.
Assume that $g(e_i) = g(e_p)$
$g\big(v_{(m-2)n+i}, v_{(m-1)n+i}\big) = g\big(v_{(m-2)n+p}, v_{(m-1)n+p}\big)$
$f\big(v_{(m-2)n+i}\big)f\big(v_{(m-1)n+i}\big) = f\big(v_{(m-2)n+p}\big)f\big(v_{(m-1)n+p}\big)$
$[2i+(m-3)n][(m-1)n+i] = [2p+(m-3)n][(m-1)n+p]$
$i = \frac{5n-3mn}{2} - p$, a contradiction for i.
Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i,p \leq n$.
Hence all the induced edge labeling in E_2 are distinct.
We shall show that the labelings in edge set E_3 are distinct.
Define an edge induced function $g : E_3 \to N$ such that
$g(e_i) = f(v_{(m-1)n+i})f(v_{nm+1})$, for all $e_i \in E_3$.
If e_i and e_p are distinct edges in E_3 then to prove $g(e_i) \neq g(e_p)$.
For $i < p, 1 \leq i,p \leq n, j,q = m-1$.
Assume that $g(e_i) = g(e_p)$
$g\big(v_{(m-1)n+i}, v_{nm+1}\big) = g\big(v_{(m-1)n+p}, v_{nm+1}\big)$
$f\big(v_{(m-1)n+i}\big)f(v_{nm+1}) = f\big(v_{(m-1)n+p}\big)f(v_{nm+1})$
$[(m-1)n+i][mn+1] = [(m-1)n+p][mn+1]$
$\Rightarrow i = p$, a contradiction for i.

Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i,p \leq n, j,q = m-1$.
Hence all the induced edge labeling in E_3 are distinct.
similarly all edge labeling in E_1 & E_2 and E_2 & E_3 can be proved to be distinct as in the case for $m-$ even. Thus all the edge labeling of E_1, E_2 and E_3 are distinct. Also the induced edge labeling of E_1 & E_2 and E_2 & E_3 are distinct.
Hence the Mongolian Network has strongly multiplicative labeling for $m-$ odd any n.

Hence Mongolian Network is Strongly Multiplicative for all n and m.

4. CONCLUSION

In this paper we have proved that Double Circus $DCT(n)$ for all n and Mongolian networks $M_{m,n}$ for all n and m are strongly multiplicative. Finding strongly multiplicative labeling for other interconnection networks is quite challenging.

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