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K-Clean Rings

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ABSTRACT

Let R be an associative ring with identity. R is called a clean ring if every element of R is the sum of an idempotent and a unit. So far many types of clean rings is introduced. In this article we define K-clean rings as extending of 2-good rings, and study it's properties, and express the result of it. So show under what conditions semilocal rings and semiperfect rings are K-clean ring. Finally, we define strongly K-clean ring and study it's properties.

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1. INTRODUCTION

Clean rings were introduced by Nicholson[13]. An element r in a ring R is said to be clean if it can be written as the sum of a unit and an idempotent in R. A ring R is called a clean ring if every element of R is clean. Semiperfect and unit-regular rings are examples of clean rings as shown by Camillo and Yu[10] and Camillo and Khurana[8]. So far many types of clean rings is introduced such that it can be mentioned semiclean rings[7], n-clean rings[3], g(x)-clean rings[6] and G-clean rings[4]. In recent years, many authors have studied clean rings and their generalizations such as.

Definition 1.1. An element $x \in R$ is said to be a full element if there exist $s; t \in R$ such that $sxt = 1$. The set of all full elements of a ring R will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$.[1]

A ring is called n-good[2] if every it's element is a sum of n units. In this paper, we extend 2-good rings and introduce the concept of K-clean rings. Examples of K-clean rings are given. We study various properties of K-clean rings, and prove that $M_n(R)$ is K-clean for any K-clean ring R. An element is a ring R is strongly clean if it is the sum of an idempotent and a unit that commute. A ring R is strongly clean if every element of R is strongly clean. Local rings are obviously strongly clean ring. We define strongly K-clean ring and check it's primary properties. We use $U(R)$, $Id(R)$ and $J(R)$ to denote the group of units, set of idempotent elements and Jacobson radical of R, respectively, and will write $\bar{R} = R/J(R)$. The $n \times n$ upper triangular matrix ring over R is denoted by $Tn(R)$.

2. K-Clean RING AND IT'S PROPERTIES

Definition 2.1. An element r in a ring R is said to be *K-clean* if it can be written as the sum of a unit and an full element in R. A ring R is called a K-clean ring if every element of R is K-clean.

Lemma 2.2. The following statements hold for any ring R.

(1) If $w \in K(R)$, then uw and wu are both in $K(R)$ for any $u \in U(R)$.

(2) $w \in K(R)$ if and only if $\bar{w} \in K(\bar{R})$.

(3) For any $e \in Id(R)$, $K(eRe) + K((1-e)R(1-e)) \subseteq K(R)$.

Proof. Proof of (1), (2) is in ([1], lemma 3.1). For (3), let $x \in K(eRe)$ and $y \in K((1-e)R(1-e))$. Then there exist $s, t \in eRe$ and $s', t' \in (1-e)R(1-e)$ so that $sxt = e, s'yt' = 1-e$.

Hence we have

$$(s+s')(x+y)(t+t') = sxt + sxt' + s'xt + s'xt' + syt + syt' + s'yt + s'yt' = e + (1-e) = 1$$

Therefore (3) so is hold. □

Yu call a ring R to be a left quasi-duo ring if every maximal left ideal of R is a two-sided ideal. commutative rings and local rings are belong to this class of rings.

Lemma 2.3. *For a left quasi-duo ring R, the following statements are equivalent:*

(1) R is a 2-good ring.

(2) R is a K-clean ring.

Proof. It is similar to ([1], Theorem 2.9). □

Example 2.4. Every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is K-clean[14]. The endomorphism ring of a complete module over a complete discrete valuation ring is K-clean[12]. Let M be a free R–modules where R is a *PID*, if the rank of M is finite and greater than 1, then $end_R(M)$ is K-clean[9]. The endomorphism ring of a free R–module of rank at least 2 in K-clean, where R is a *PID*[5].

Proposition 2.5. *Any homomorphic image of a K-clean ring is K-clean.*

Proof. Let R be a K-clean ring and let $f: R \to S$ be a ring surjective homomorphism. Then for any $y \in S$, there exist $x \in R$ such that $f(x) = y$. Since R is K-clean, therefore $x = w + u$ with $w \in K(R)$ and $u \in U(R)$ so that exist $s, t \in R$ with $swt = 1$. Hence $f(x) = f(w) + f(u)$. Obviously $f(u) \in U(S)$ and $1 = f(1) = f(swt) = f(s)f(w)f(t)$.

Hence the proof is complete.

Proposition 2.6. *A _nite direct product $R = \prod_{i=1}^{n} Ri$ of rings Ri is K-clean if and only if each Ri is K-clean.*

Proof. If R is K-clean, then each Ri is K-clean by proposition 2.5.

Conversely, assume each R_i is K-clean, and $x = (x_i) \in R$. Then $x_i = w_i+u_i$ with $w_i \in K(R_i)$ and $u_i \in U(R_i)$ for each i. We can identify R_i with $(\ldots, 0, R_i, 0, \ldots)$ canonically. Let $e_i = (\ldots,0, 1, 0,\ldots)$. Then we have
$(w_i) = (w_1, 0,\ldots, 0) + (0,w_2,\ldots,0) +\ldots+ (0, 0, \ldots,w_n) \in$
$K(e_1Re_1) + K(e_2Re_2) + \ldots+ K(e_nRe_n) \subseteq K(R)$.
Now
$x = (x_i) = (w_i + u_i) = (w_i) + (u_i)$, with $(w_i) \in K(R)$ and $(u_i) \in U(R)$, so R is a K-clean ring.

Proposition 2.7. *The ring R is K-clean if and only if the ring* $R[[x]]$ *of formal power series over R is K-clean.*

Proof. If $R[[x]]$ is K-clean, then R is K-clean by proposition 2.5. Now if R is K-clean, then for any $f(x)$ 2 $R[[x]]$,
$f(x) = a_0 + a_1x +\ldots + a_nx^n +\ldots$.
By assumption, $a_0 = w + u$ with $w \in K(R)$ and $u \in U(R)$. Hence
$f(x) = w + u + a_1x + \ldots + a_nx^n + \ldots$ with $w \in K(R) \subseteq K(R[[x]])$ and $u + a_1x +\ldots + a_nx^n +\ldots\in U(R[[x]])$, as desired.

Lemma 2.8. *Let* $e \in Id(R)$ *be such that eRe and* $(1 - e)R(1 - e)$ *are both K-clean rings. Then R is a K-clean ring.*

***Proof*.** For convenience, write $r = 1 - r$ for each $r \in R$. By use the Pierce decomposition of the ring R, we have:
$R = eRe + eR\bar{e} + \bar{e}Re + \bar{e}R\bar{e}$.
Now let $x = a+b+c+d$ where $a \in eRe$, $b \in eR\bar{e}$, $c \in \bar{e}Re$ and $d \in \bar{e}Re$.
By hypothesis, write $a = w + u$ where $w \in K(eRe)$ and $u \in U(eRe)$ with inverse u_1. Then $d-cu_1b \in \bar{e}R\bar{e}$, so write $d-cu_1b = y + v$ where $y \in K(\bar{e}R\bar{e})$ and $v \in U(\bar{e}R\bar{e})$ with inverse v_1. Hence $x = (w +y)+u+ b + c + v + cu_1b$ and it suffices to show that $u + b + c + v + cu_1b$ is a unit in R. To this end compute

$(u+b+c+v+cu_1b)(u_1+u_1bv_1cu_1-u_1bv_1-v_1cu_1 + v_1) = (e+bv_1cu_1- bv_1)$
$+(-bv_1cu_1 + bv_1) + (cu_1 + cu_1bv_1cu_1 - cu_1bv_1)$
$+ (-cu_1 + 1 - e) +(-cu_1bv_1cu_1 + cu_1bv_1) = 1$.

Similarly, $(u_1+u_1bv_1cu_1-u_1bv_1-v_1cu_1+v_1)$
$(u+b+c+v+cu_1b) = 1$.
Note that $w + y \in K(eRe) + K(\bar{e}R\bar{e}) \subseteq K(R)$. Therefore proof is complete.

Using Lemma 2.8, an inductive argument gives immediately.

Theorem 2.9. *If* $1 = e_1 +\ldots+e_n$ *in a ring R where* e_i *are orthogonal idempotents and for each ei, eiRei is K-clean, then R is K-clean.*

The following two results are direct consequences of Theorem 2.9:

Corollary 2.10. *If R is a K-clean ring, then also is the matrix ring Mn(R).*

Corollary 2.11. *If* $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ *are modules and end*(M_i) *is K-clean for each i, then end(M) is K-clean.*

3. K-Clean AND KSR RINGS

Definition 3.1. Let $a, b \in R$ satisfy $aR + bR = R$, then R is called KSR, if exist $w \in K(R)$ such that $a + bw \in U(R)$.[1]

Proposition 3.2. *Let R be a ring. Then the following statements are equivalent:*

(1) R is KSR.
(2) Whenever $a, b \in R$ *satisfy* $ax + b = 1$*, there exist* $w \in K(R)$ *such that* $a + bw \in U(R)$.

(3) Whenever a, b ∈ R satisfy ax + b = 1, there exist y ∈ R such that a + by ∈ U(R) and $1 - xy \in K(R)$.([1], Proposition 3.3)

Corollary 3.3. *Let R is a KSR ring, then R is K-clean.*

Proof. Since $1 = x + (1 - x)$ for any $x \in R$ and R is KSR ring, so there exist $w \in K(R)$ such that $x + w \in U(R)$. Hence the proof is complete. □

Lemma 3.4. *Let R be a ring. Then R is KSR ring if and only if so does* $\bar{R}$.([1], Proposition 3.5)

Example 3.5. Let R be a local ring with $2 \notin J(R)$. Then R is K-clean. ([1], Example 3.6)

Lemma 3.6. *Let R be a ring. Then R is K-clean ring if and only if so does* $\bar{R}$.

Proof. It is clear by Lemma 2.2 and ([11], Proposition 4.8). □

Lemma 3.7. [1] *Let R be any left semisimple ring. Then* $R \cong M_{n1}(D_1) \times \dots \times M_{nr}(D_r)$ *for suitable division rings* $D_1, \dots, D_r$ *and positive integers* $n_1, \dots; n_r$. *The number r is uniquely determined, as are the pairs* $(n_1, D_1), \dots, (n_r; D_r)$ *(up to a permutation)*.([11], Theorem 3.5).

Theorem 3.8. *Let R has not any factor of* $M_n(\mathbb{Z}_2)$ *where* $n \geq 1$. *Then every left semisimple ring R is a K-clean ring.*

Proof. Since for any division ring $D \neq \mathbb{Z}_2$, $2 \in U(R)$, so $M_n(D)$ is K-clean. See example 3.5, proposition 2.6 and corollary 2.10. □

Corollary 3.9. *Let R/rad R has not any factor of* $M_n(\mathbb{Z}_2)$ *where* $n \geq 1$. *Then the following statements is hold:*

(1) *Every semiperfect ring R is a K-clean ring.*
(2) *Every semilocal ring R is a K-clean ring.*

Proof. Proof is clear by theorem 3.8 and lemma 3.6. □

4. STRONGLY K-Clean RING

Definition 4.1. An element r of a ring R is called strongly K-clean if $r = w + u$, where $w \in K(R)$, and u is unit in R such that $wu = uw$. A ring R is called a strongly K-clean ring if every element of R is strongly K-clean.

Theorem 4.2. *(1) Every homomorphic image of a strongly K-clean ring is strongly K-clean.*
(2) A direct product $R = \prod_{i=1}^{n} R_i$ *of rings* R_i *is a strongly K-clean ring if and only if each* R_i *is a strongly K-clean ring.*

Proof. (1), (2) are obvious from proposition 2.5, proposition 2.6. □

Corollary 4.3. *a ring R is strongly K-clean if and only if for any ideal I of R, R/I is strongly K-clean.*

Corollary 4.4. *Let R be a ring and* $1 < n \in \mathbb{N}$. *If* $T_n(R)$ *is strongly K-clean ring, then R is strongly K-clean.*

Proof. Let $A = (a_{ij}) \in T_n(R)$ with $a_{ij} = 0$ and $1 \leq j \leq i \leq n$. Note that $\phi : T_n(R) \rightarrow R$ with $\phi(A) = a_{11}$ is a ring epimorphism. □

Corollary 4.5. *Let R be a ring. If the formal power series ring R[[t]] is strongly K-clean, then R is strongly K-clean.*

Proof. This is because $\psi : R[[t]] \rightarrow R$ with $\psi(f) = a_0$ is a ring epimorphism where

$f = \Sigma_{i \geq 0} a_i t_i \in R[[t]]$. □

Proposition 4.6. *If* $e^2 = e \in R$ *and* $r \in eRe$ *is strongly K-clean in eRe and* $(1-e) \in K((1-e)R(1-e))$*, then r is strongly K-clean in R.*

Proof. Since r is strongly K-clean in eRe, there exist a w 2 $K(R)$ and a unit u in eRe such that $r = w + u$, $wu = uw$ and $uy = yu = e$ for $y \in eRe$. Then $v = u-(1-e)$ is a unit in R (with $v^{-1} = y - (1-e)$) and $r-v = w+(1-e)$. Since $w \in K(eRe)$ and $(1-e) \in K((1-e)R(1-e))$, Hence $r - v \in K(R)$ and $v(r-v) = [u-(1-e)][w+(1-e)] = uw+0+0-(1-e)= wu-(1-e) = [w + (1 - e)][u - (1 - e)] = (r - v)v$.
Therefore r is strongly K-clean in R. □

Theorem 4.7. *Let e is the central idempotent of R. For an element* $x \in R$*, x is strongly K-lean in R if and only if ex is strongly K-clean in eRe and* $(1 - e)x$ *is strongly K-clean in* $(1-e)R(1- e)$*.*

Proof. (⇒). Since x is strongly K-clean in R, then $x = w + u$ where$w \in K(R)$, $u \in U(R)$ and $wu = uw$. It is easy to say that $ew \in K(eRe)$, $eu \in U(eRe)$ and $(ew)(eu) = (eu)(ew)$. Thus $ex = ew + eu$ is strongly K-clean. Similarly, $(1-e)x$ is strongly K-clean in $(1-e)R(1-e)$.
(⇐). Since ex is strongly K-clean in eRe and $(1- e)x$ is strongly K-clean in $(1- e)R(1- e)$, then $ex = f_1 + u_1$, $(1- e)x = f_2 + u_2$ where $f_1 \in K(eRe)$, $f_2 \in K((1- e)R(1- e))$ and $u_1 \in U(eRe)$, $u_2 \in U((1- e)R(1- e))$ and $f_1 u_1 = u_1 f_1, f_2u_2 = u_2f_2$.

Let $f = f_1 + f_2$, then $f \in K(R)$. Since $u_1v_1 = e = v_1u_1$, $u_2v_2 = (1- e) = v_2u_2$ and $u_1v_2 = 0 = u_2v_1$, then $(u_1 + u_2)(v_1 + v_2) = 1 =(v_1 + v_2)(u_1 + u_2)$. So $u = u_1 + u_2$ is unit in R. Furthermore we have $fu = (f_1+f_2)(u_1+u_2) = f_1u_1+f_2u_2 = u_1 f_1+u_2f_2 = (u_1+u_1)(f_1+f_2) = uf$, and $f + u = f_1 + f_2 + u_1 + u_2 = ex + (1 - e)x = x$.
The proof is complete. □

Corollary 4.8. *Let e is a central idempotent of R. R is strongly K-clean if and only if eRe and* $(1-e)R(1-e)$ *are both strongly K-clean.*

Corollary 4.9. *For an element* $x \in R$*. Let* $1 = e_1+e_2+\ldots+e_n$ *in* $R, n > 1$*, where* e_i *are orthogonal central idempotents. Then x is strongly K-clean in R if and only if* e_ix *is strongly K-clean in* e_iRe_i *for each i.*

Corollary 4.10. *Let* $1 = e_1 + e_2 + \ldots + e_n$ *in R,* $n > 1$*, where* e_i*'s are orthogonal central idempotents. R is strongly K-clean if and only if each* e_iRe_i *is strongly K-clean.*

A property P of rings is said to be *Morita invariant* if whenever two rings R and S have module categories which are categorically equivalent, then R has property P if and only if S has property P. It is well known that a property P is Morita invariant if and only

1. If R has P, and $n \in \mathbb{N}$, then $M_n(R)$ has P, and
2. if R has P, and $e \in R$ is a full idempotent (i.e. $ReR = R$), then eRe has P.

Question1: Is strongly K-clean a Morita invariant?

Question2: What is relation between strongly clean and strongly K-clean?
In case latter question, we can express this:

Proposition 4.11. *If R is strongly clean and* $2 \in U(R)$*, then R is strongly K-clean.*

Proof. Let R is strongly clean and $x \in R$. Then $\frac{1}{2}(x+1) = e+u$ where $e^2 = e$, $u \in U(R)$ and $eu = ue$, therefore $x = 2u + (2e - 1)$. We can view that $2u \in U(R)$, $(2e-1) \in K(R)$ and $(2u)(2e-1) = (2e-1)(2u)$, easily. Hence x is strongly K-clean. □

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