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Valuation on Some Algebric Structurs

M. H. Hosseini and M. H. Rezaeigol

Academic Member of School of Mathematics,
University of Birjand, IRAN.

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ABSTRACT

The purpose of this article is to introduce the notion of valuations on Some algebraic structures; valuation on $M(R)$ (the set of square matrices), on $M^{(3)}(R)$ (set of cubic matrices) over the field P, valuation on simpleartinian rings, semisimple rings and finally valuation on a module overa commutative ring and their some basic results.

Keywords: valuation ; matrix valuation;cubic matrix; semisimple ring; valuation on module.

1 INTRODUCTION

In what follows, R and $M(R)$ will represent, an associative ring with unit and the set of all square matrices respectively.

Definition 1.1.(see [1]) A valuation on ring R is a function $\vartheta : R \rightarrow \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian additive group such that for all $a, b \in R$,

1) $\vartheta(ab) = \vartheta(a) + \vartheta(b)$;
2) $\vartheta(a + b) \geq min\{\vartheta(a), \vartheta(b)\}$;
3) $\vartheta(a) = \infty$ if and only if $a = 0$.

One can easily show that, $\vartheta(1)=0$ and $\vartheta(-a) = \vartheta(a)$

The first generalization for valuation rings of division rings was obtained by Schilling in[6].

Remark 1.3. If $\Gamma = Z$, then valuation onskewfield K is called a discrete valuation.

Example 1.2. Let p be a prime number. Map $\vartheta : Q \rightarrow Z \cup \{\infty\}$, by $\vartheta(p^r m/n) = r$, where $r \geq 0$ and p divides neither m nor n, is a discrete valuation.

Also if $K = k(x)$, where k is any field and R be the set of all rational functions $p^r m/n$, where $r \geq 0, p$ is a fixed polynomial that is irreducible over k and m and n are arbitrary polynomials in $k[x]$ not divisible by p. Then map

$\vartheta : K \rightarrow Z \cup \{\infty\},$

$by \ \ v(f/g) = deg\,(g) - deg\,(f)$

is a discrete valuation.

2. MATRIX VALUATION

Let $M(R)$is the Set ofall square matrices over ring R. For any two matrices$A$ and B over R we define the diagonal sum of A and B as:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Given two matrices

$A = (\ A_{1,}A_{2,}A_{3,}\ \dots, A_{n,})$and

$B= (\ B_{1,}A_{2,}A_{3,}\ \dots, A_{n,})$ in $M(R)$,The determinant sum of $A$and B, with respect to the first column, is defined by

$$A \nabla B = (\ A_1 + B_{1\,,}A_{2,}A_{3,}\ \dots, A_{n,})$$

Definition 2.1.(see [3])**.** A valuation (or a classical valuation)on the set of all square matrices $M(R)$ over a skew field R is a mapping

$|\ |: M(R) \rightarrow \ \Gamma \ \cup \ \{\infty\},$

satisfying the following conditions:

(1) if$A \ \in M_n(R)$and $r(A) < \ n,$ then $|A| \ = \ \infty;$

(2) if$A, B \ \in M_n(R), 1 \leq \ j \leq n, Bj = -Aj$, and $Bi = Ai \ for \ i \neq j, 1 \leq i \ \leq n,$ then $|A| \ = \ |B|;$

(3) if$A, B \ \in M_n(R)$,which exists the determinantal sum $A\nabla B$, then $|A\nabla B| \ \geq \ min\,\{|A|, |B|\};$

(4) for any $A \in M_n(R) \ \ and \ B \in M_m(R),$ $|A \ \oplus \ B| = \ |A| + \ |B|;$

(5) $|I| = \ 0,$ where I is an identity matrix.

***Proposition* 2. 2.** Let $\vartheta : \ R \ \rightarrow \ \ \Gamma \ \cup \{\ \infty\}$be a valuation on commutative ring R. mapping

$V \ : \ M(R) \rightarrow \Gamma \ \ \cup \{\ \infty\},$

by $V\ (A) = \ \vartheta(detA)$ is a matrix valuation.

***Proposition* 2. 3.** Let R be a division ring with an abelian valuation ν. Then ν may be extended to a matrix valuation V on $A \ = \ M_n(R)$, for each $n \ \geq \ 1$ by the equation:

$$V(X) = \ \nu(DetX), X \ \epsilon \ A,$$

where”Det” denotes the Dieudonne determinant, together with the rule $V(X) \ = \ \infty$ when X is singular.

Proof. (See [5]).

Corollary 2.4. The correspondence $\nu \ \leftrightarrow \ V$ in Proposition 2.2 andProposition 2.3 is a bijection between abelian valuations on R and matrix valuations on$A \ = \ M_n(R)$.

3. VALUATIONS ON THE SET OF CUBIC MATRICES

Let F be a field and $M^{(3)}(F)$ be the set of all cubic matrices over the field F. All necessary definitions and notations on cubic matrices can be found in (see [7, Chaps. I, II]).

Definition 3.1.A valuation on the set of all cubic matrices $M^{(3)}(F)$ over a field F is a mapping

$\mu: M^{(3)}(F) \rightarrow \ \Gamma \ \cup \ \{\infty\},$

satisfying the following conditions:

(1)if$A \ \in M^{(3)}(F)$ and $r(A) < \ n,$ then $\mu(A) \ = \ \infty;$

(2)if $A, B \in M^{(3)}(F), 1 \leq j \leq n, Bj = -Aj$, and $Bi = Ai$ for $i \neq j, 1 \leq i \leq n$, then $\mu(A) = \mu(B)$;
(3)if $A, B \in M^{(3)}(F)$, which exists the determinantal sum $A\nabla B$, then $\mu(A\nabla B) \geq min\{\mu(A), \mu(B)\}$;
(4) for any $A \in M_n(F)$ and $B \in M_m(F)$, $\mu(A \oplus B) = \mu(A) + \mu(B)$;
(5) $|I| = 0$, where I is an identity matrix.

Remark 3.2. Proposition 2.2, proposition 2.3 and corollary 2.4, similarly is established for the cubic matrices ring over a field.

4. VALUATIONS ON SEMISIMPLE RINGS

In this section, we show that a valuation on simple artinian ring[5] may be extended to a Valuation on semisimple ring by Wedderburn-artin theorem.

Theorem 4.1.(Wedderburn-artin). Let R be any left semisimple ring. Then

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots \times M_{n_r}(D_r)$$

for suitable division rings $D_1, D_2, \ldots, D_r$ and positive integers $n_1, n_2, \ldots, n_r$ (up to a permutation).there are exactly mutually non isomorphic leftsimple module over R[4].

Now let $D_1, D_2, \ldots, D_r$ are division rings with valuations ϑ_i on D_i (for i =1, 2,· · ·, r). Then there exists simple valuations μ_i on $A_i = M_{n_i}(D_i)$ (by proposition 2) such that for any $X_i \in A_i, \mu_i(X_i) = \vartheta_i(DetX_i)$. Therefore we have :

Theorem 4.2. Map μ defined by $\mu(X_1, X_2, \ldots, X_r) = (\mu_1(X_1), \mu_2(X_2), \ldots, \mu_r(X_r))$ is a valuation on semisimple ring

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots \times M_{n_r}(D_r)$$

5. VALUATION ON MODULES

In this section, the first we defined a valuations on a module, and show that if M be a R-module, then for any valuation on M exist a *Manis valuation on the ring* R. Moreover in case $R = Z$, one can show that any classic valuation on Z, under a certain condition, is a valuation on Z-mod$M$.

Definition 5.1.[2] Let M be an R-module where R is a ring and Γ be an orderedset with maximum element ∞. A mapping μ of M ontocalled a valuation on M, if the following conditions are satisfied:
(i) For any $x, y \in M, \mu(x + y) \geq min\{\mu(x), \mu(y)\}$;
(ii) If $\mu(x) \leq \mu(y), x, y \in M$, then $\mu(ax) \leq \mu(ay)$ for all $a \in R$;
(iii) Put $\mu^{-1}(\infty) = \{x \in M | \mu(x) = \infty\}$. If $\mu(az) \leq \mu(bz)$, where $a, b \in R$, and $z \in M\backslash\mu^{-1}(\infty)$, then $\mu(ax) \leq \mu(ay)$ for all $x \in M$;
(iv) For every $a \in R\backslash(\mu^{-1}(\infty) : M)$, there is an $a' \in R$ such that $\mu((a'a)x) = \mu(x)$. In this case Γ is called the value set of μ, and $\mu^{-1}(\infty)$ is called the core of μ.

Definition 5.2 $\mu^{-1}(\infty)$ is called the *core* of μ.

***Remark* 5.3.** For an ordered abelian group M and an ordered set Γ with maximum element ∞, a mapping μ of M onto Γ is defined to be a valuation on M, if the following conditions are satisfied:

(1) For $x \in M, \mu(x) = 1$ if and only if $x = 0$;
(2) For any $x, y \in M, \mu(x + y) \geq min\{\mu(x), \mu(y)\}$;
(3) For every nonzero integers $n, \mu(nx) = \mu(x)$.

Viewing M as a Z −module, it is easy to see that such a mapping μ is a valuationonM in the sense of valuations on a module. In this case, the coreofμ is $\{0\}$, andthe induced valuation pair is $(M, \{0\})$.

Example 5.4.. Let V be a nonzero vector space over a field F (i.e.,a Fmodule) with base B and let w be an arbitrary valuation on F with value group Γ. For every nonzero α∈ V,α may be uniquely expressed as

$$\alpha = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$where\ x_1, x_2, \ldots, x_n \in \mathcal{B}\ and\ a_1, a_2, \ldots, a_n \in F\backslash\{0\}$.
Then mapping

$$\vartheta : V \rightarrow = \mathcal{B} \times \Gamma \cup \{\infty\}$$

By$\vartheta(\alpha) = (x_1, \omega(a_1))$ and $\vartheta(0) = \infty\ is\ a\ valuation\ on\ V\ as\ a\ F - module\ with\ core$ $\{0\}$.

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