

# Common Fixed Point Result in Dislocated Quasi Metric Spaces

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(Received on: December 14, 2017)

## ABSTRACT

In this paper, a common fixed point theorem of a pair of self-maps is proved without appeal to continuity condition in dislocated quasi metric spaces. It extends and generalizes the result of Sarma *et al.*<sup>7</sup>, Theorem 5 to two self-maps by applying a generalized contraction. It further unifies the result of Dubey *et al.*<sup>4</sup>, Theorem 3.1 and some well-known fixed point results in the literature.

**AMS Subject Classification:** 47H10, 54H25.

**Keywords:** Complete dislocated quasi-metric, Contraction, Common fixed point.

## INTRODUCTION

Dislocated topologies serves as an essential tool in view of its utility in the pursuit of developing logic programming (see<sup>5,6</sup>). In 2000, Hitzler and Seda<sup>6</sup> proved a fixed point theorem in complete dislocated metric spaces as a generalization of the celebrated Banach contraction principle.

In 2006, Zeyada *et al.*<sup>9</sup> initiated the notion of complete dislocated quasi-metric space as a generalization of dislocated metric space, and generalized the result of Hitzler *et al.*<sup>6</sup> in such space. In 2008, Aage and Salunke<sup>1</sup> proved two fixed point theorems of continuous map in complete dislocated quasi-metric space which generalized the result of Zeyada *et al.*<sup>9</sup>. In the same year, the same authors proved fixed point theorems for Kannan and Ciric types of maps in complete dislocated quasi-metric space (see<sup>2</sup>). Afterwards, a few papers dealt with fixed points in such space were obtained (for instance<sup>7,8</sup> etc).

In 2014, Sarma *et al.*<sup>7</sup>, Theorem 5 improved the result of Aage and Salunke<sup>2</sup>, Theorem 3.3 by dropping continuity condition, stated below as Theorem 1.1.

**Theorem 1.1.** Let  $(X, d)$  be a complete dq-metric space, and let  $T : X \rightarrow X$  be a self-map satisfying the following condition:

$$d(Tx, Ty) \leq a \{d(x, Tx) + d(y, Ty)\}$$

for all  $x, y \in X$ , where  $0 \leq a < \frac{1}{2}$ .

Then  $T$  has a unique fixed point in  $X$ .

The objective of this paper is to extend and generalize Sarma *et al.*<sup>7</sup>, Theorem 5 to two self-maps by applying a generalized contraction, and then to unify the results of Dubey *et al.*<sup>4</sup>, Theorem 3.1 and Aage *et al.*<sup>1</sup>, Theorem 3.1.

Throughout this paper,  $\mathbb{N}$  denotes the set of positive integers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

## 2. PRELIMINARIES

We need to retrieve the following relevant definitions and results in the sequel.

**Definition 2.1.** ([9]). Let  $X$  be a non-empty set and let  $d : X \times X \rightarrow [0, \infty)$  be a function satisfying the following conditions:

- (i)  $d(x, y) = d(y, x) = 0$  implies  $x = y$
- (ii)  $d(x, y) \leq d(x, z) + d(z, y)$ , for all  $x, y, z \in X$ .

Then  $d$  is called a dislocated quasi-metric (in short, dq-metric) on  $X$ , and the pair  $(X, d)$  is called a dislocated quasi-metric space (in short, dq-metric space).

In addition, if  $d$  satisfies  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , then it is called a dislocated metric.

A metric on a set is an example of dislocated metric which is also a dislocated quasi metric, but a dislocated quasi-metric is not necessarily dislocated metric and so it is not a metric.

A simple illustration of these facts is furnished in the following.

**Example 2.2.** Let  $X = [0, 1]$ . Define  $d : X \times X \rightarrow [0, \infty)$  by  $d(x, y) = |x - y| + |x|$  for all  $x, y \in X$ . Then  $d$  is a dislocated quasi-metric space on  $X$ , but symmetric condition fails to hold and therefore, it is neither dislocated metric nor metric on  $X$ . In what follows,  $X$  denotes dislocated quasi-metric space  $(X, d)$ .

**Definition 2.3.** ([9]). A sequence  $\{x_n\}$  in dq-metric space  $X$  is called dq-convergent if for  $n \in \mathbb{N}$ ,  $\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$ .

In this case,  $x$  is called a dislocated quasi limit (in short, dq-limit) of the sequence  $\{x_n\}$ .

**Lemma 2.4.** ([9]). dq-limits in a dq-metric space are unique.

**Lemma 2.5.** ([1]). Every subsequence of dq-convergent sequence to a point  $x_0$  is dq-convergent to  $x_0$ .

**Definition 2.6. ([9]).** A sequence  $\{x_n\}$  in dq-metric space  $X$  is called Cauchy sequence if for each  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $d(x_m, x_n) < \varepsilon$  or  $d(x_n, x_m) < \varepsilon$  for all  $m, n \geq n_0$ .

**Definition 2.7. ([9]).** A dq-metric space  $X$  is called complete if every Cauchy sequence in it is dq-convergent.

In<sup>3</sup>, Ciric generalized the Banach contraction principle in the context of metric space.

**Definition 2.8. ([3]).** Let  $(X, d)$  be a metric space. A map  $T : X \rightarrow X$  is called a generalized contraction if and only if there exist non-negative numbers  $a_1, a_2, a_3$  and  $a_4$  satisfying  $\sup \{ a_1 + a_2 + a_3 + 2a_4 \} < 1$  such that for every  $x, y \in X$ ,

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 \{ d(x, Ty) + d(y, Tx) \}$$

### 3. MAIN RESULT

**Theorem 3.1.** Let  $(X, d)$  be a complete dq-metric space, and let  $S, T : X \rightarrow X$  be a pair of self-maps satisfying the following condition:

$$d(Sx, Ty) \leq a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 \{ d(x, Ty) + d(y, Sx) \} \quad \dots \quad (3.1)$$

for all  $x, y \in X$ , where  $a_i \geq 0$  with  $a_1 + a_2 + a_3 + 4a_4 < 1$ .

Then  $S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof.** Let us choose  $x_0 \in X$  arbitrary. We define a sequence  $\{x_n\}$  in  $X$  such that

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1} \text{ for all } n \in \mathbb{N}_0.$$

We consider  $d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$ .

In view of (3.1), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, Sx_{2n}) + a_3 d(x_{2n+1}, Tx_{2n+1}) \\ &\quad + a_4 \{ d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n}) \} \\ &= a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) \\ &\quad + a_4 \{ d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) \} \\ &\leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) \\ &\quad + a_4 \{ d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \} \end{aligned}$$

$$\begin{aligned}
 &= (a_1 + a_2 + 2a_4)d(x_{2n}, x_{2n+1}) + (a_3 + 2a_4)d(x_{2n+1}, x_{2n+2}) \\
 \Rightarrow &(1 - a_3 - 2a_4)d(x_{2n+1}, x_{2n+2}) \leq (a_1 + a_2 + 2a_4)d(x_{2n}, x_{2n+1}) \\
 \Rightarrow &d(x_{2n+1}, x_{2n+2}) \leq \left( \frac{a_1 + a_2 + 2a_4}{1 - a_3 - 2a_4} \right) d(x_{2n}, x_{2n+1}) \\
 \Rightarrow &d(x_{2n+1}, x_{2n+2}) \leq \lambda d(x_{2n}, x_{2n+1}), \text{ where } \lambda = \frac{a_1 + a_2 + 2a_4}{1 - a_3 - 2a_4} < 1.
 \end{aligned}$$

Similarly, we have  $d(x_{2n}, x_{2n+1}) \leq \lambda d(x_{2n-1}, x_{2n})$ .

So, we obtain  $d(x_{2n+1}, x_{2n+2}) \leq \lambda^2 d(x_{2n-1}, x_{2n})$ .

Proceeding in this way, we have  $d(x_{2n+1}, x_{2n+2}) \leq \lambda^{2n+1} d(x_0, x_1)$ .

We claim that  $\{x_n\}$  is a Cauchy sequence in  $X$ .

Now, for  $n, k \in \mathbb{N}$ , we see that

$$\begin{aligned}
 d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \dots + d(x_{n+k-1}, x_{n+k}) \\
 &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \dots + \lambda^{n+k-1}) d(x_0, x_1) \\
 &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \dots) d(x_0, x_1) \\
 &= \left( \frac{\lambda^n}{1 - \lambda} \right) d(x_0, x_1).
 \end{aligned}$$

Since  $\lambda < 1$ ,  $\lambda^n \rightarrow 0$  as  $n \rightarrow \infty$  and so,  $d(x_n, x_{n+k}) \rightarrow 0$ . Similarly, we can show that  $d(x_{n+k}, x_n) \rightarrow 0$ . Thus,  $\{x_n\}$  is a Cauchy sequence in  $X$ . It follows that completeness of  $X$  implies existence of  $u \in X$  such that  $\lim_{n \rightarrow \infty} d(x_n, u) = \lim_{n \rightarrow \infty} d(u, x_n) = 0$ . Also the subsequences  $\{x_{2n+1}\}$  and  $\{x_{2n+2}\}$  of the sequence  $\{x_n\}$  converge to  $u$ .

Now, we claim that  $Su = Tu = u$ .

We have  $d(u, Su) \leq d(u, x_{2n}) + d(x_{2n}, Su)$

$$= d(u, x_{2n}) + d(Tx_{2n-1}, Su)$$

By using (3.1), we have

$$\begin{aligned}
 d(u, Su) &\leq d(u, x_{2n}) + a_1 d(x_{2n-1}, u) + a_2 d(x_{2n-1}, Tx_{2n-1}) + a_3 d(u, Su) \\
 &\quad + a_4 \{d(x_{2n-1}, Su) + d(u, Tx_{2n-1})\}
 \end{aligned}$$

$$= d(u, x_{2n}) + a_1 d(x_{2n-1}, u) + a_2 d(x_{2n-1}, x_{2n}) + a_3 d(u, Su) + a_4 \{d(x_{2n-1}, Su) + d(u, x_{2n})\}$$

Taking limit  $n \rightarrow \infty$ , we get

$$(1 - a_3 - a_4) d(u, Su) \leq 0, \text{ which is possible if } d(u, Su) = 0, \text{ since } (1 - a_3 - a_4) \neq 0.$$

Therefore,  $d(u, Su) = 0$ .

$$\begin{aligned} \text{Also, we have } d(Su, u) &\leq d(Su, x_{2n}) + d(x_{2n}, u) \\ &= d(Su, Tx_{2n-1}) + d(x_{2n}, u) \end{aligned}$$

In view of (3.1), we have

$$\begin{aligned} d(Su, u) &\leq a_1 d(u, x_{2n-1}) + a_2 d(u, Su) + a_3 d(x_{2n-1}, Tx_{2n-1}) \\ &+ a_4 \{d(u, Tx_{2n-1}) + d(x_{2n-1}, Su)\} + d(x_{2n}, u) \\ &= a_1 d(u, x_{2n-1}) + a_2 d(u, Su) + a_3 d(x_{2n-1}, x_{2n}) \end{aligned}$$

$$+ a_4 \{d(u, x_{2n}) + d(x_{2n-1}, Su)\} + d(x_{2n}, u)$$

Taking limit  $n \rightarrow \infty$ , we get  $d(Su, u) \leq (a_2 + a_4) d(u, Su)$ .

Since  $d(u, Su) = 0$ ,  $d(Su, u) \leq 0$  and so,  $d(Su, u) = 0$ . Therefore,

$$d(u, Su) = d(Su, u) = 0 \text{ and so, } Su = u.$$

Similarly, it can be shown that  $Tu = u$ .

It follows that  $Su = Tu = u$ , and therefore,  $u$  is a common fixed point of  $S$  and  $T$ .

We claim that  $u$  is the unique common fixed point of  $S$  and  $T$ .

Since  $u$  is a common fixed point of  $S$  and  $T$ , we have

$$\begin{aligned} d(u, u) &= d(Su, Tu) \\ &\leq a_1 d(u, u) + a_2 d(u, Su) + a_3 d(u, Tu) + a_4 \{d(u, Tu) + d(u, Su)\} \\ &= (a_1 + a_2 + a_3 + 2a_4) d(u, u) \end{aligned}$$

$$\Rightarrow (1 - a_1 - a_2 - a_3 - 2a_4) d(u, u) \leq 0,$$

which is possible if  $d(u, u) = 0$ , since  $1 - a_1 - a_2 - a_3 - 2a_4 \neq 0$ .

If possible, let there be another common fixed point  $v$  of  $S$  and  $T$ .

Then  $d(u, v) = d(Su, Tv)$

$$\begin{aligned} &\leq a_1 d(u, v) + a_2 d(u, Su) + a_3 d(v, Tv) + a_4 \{d(u, Tv) + d(v, Su)\} \\ &= a_1 d(u, v) + a_2 d(u, u) + a_3 d(v, v) + a_4 \{d(u, v) + d(v, u)\} \\ &= (a_1 + a_4) d(u, v) + a_4 d(v, u) \quad \dots \quad (3.2) \end{aligned}$$

Similarly, we have  $d(v, u) \leq (a_1 + a_4) d(v, u) + a_4 d(u, v) \dots (3.3)$

From (3.2) and (3.3), we have

$$|d(u, v) - d(v, u)| \leq |a_1 + a_4 - a_4| |d(u, v) - d(v, u)|$$

which implies  $d(u, v) = d(v, u)$ , since  $0 \leq a_1 < 1$ .

From (3.2), we get

$$d(u, v) \leq (a_1 + 2a_4) d(u, v), \text{ which gives } d(u, v) = 0, \text{ since } a_1 + 2a_4 < 1.$$

Further, we obtain  $d(u, v) = d(v, u) = 0$ , which implies  $u = v$ .

Hence,  $u$  is a unique common fixed point of  $S$  and  $T$ .

**Remark 3.2.** In order to valid Ciric type of contraction in dq-metric space, we employ the restriction  $a_1 + a_2 + a_3 + 4a_4 < 1$  where  $a_i \geq 0$ .

By setting  $S = T$  in Theorem 3.1, we obtain the following corollary.

**Corollary 3.3.**

Let  $(X, d)$  be a complete dq-metric space, and let  $T : X \rightarrow X$  be a self-map satisfying the following condition:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 \{ d(x, Ty) + d(y, Tx) \}$$

for all  $x, y \in X$ , where  $a_i \geq 0$  with  $a_1 + a_2 + a_3 + 4a_4 < 1$ .

Then  $T$  has a unique fixed point in  $X$ .

**Remark 3.4.** If  $a_2 = a_3$  and  $a_1 = a_4 = 0$  in Corollary 3.3, we obtain Theorem 1.1 (Sarma *et al.*<sup>7</sup>, Theorem 5) as a corollary of our Theorem 3.1.

Taking into account that  $T$  is continuous and  $S = T$  in the Theorem 3.1, we obtain the following corollary.

**Corollary 3.5.**

Let  $(X, d)$  be a complete dq-metric space, and let  $T : X \rightarrow X$  be a continuous self-map satisfying the following condition:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 \{ d(x, Ty) + d(y, Tx) \}$$

for all  $x, y \in X$ , where  $a_i \geq 0$  with  $a_1 + a_2 + a_3 + 4a_4 < 1$ .

Then  $T$  has a unique fixed point in  $X$ .

**Remark 3.6.** If  $a_2 = a_3$  and  $a_4 = 0$  in Corollary 3.5, we get Theorem 3.1 of Dubey *et al.*<sup>4</sup>.

**Remark 3.7.** If  $a_4 = 0$  in Corollary 3.5, we obtain Theorem 3.1 of Aage *et al.*<sup>1</sup>.

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