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Compounding Life Distribution – Poisson Weibull

Kumaraswamy K. and Bhatra Charyulu N.Ch.

Department of Statistics,
University College of Science, Osmania University, Hyderabad-7, Telangana, INDIA.
email: kumaraswamy@osmania.ac.in and dwarakbhat@osmania.ac.in

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ABSTRACT

In this paper a new compounding life distribution is derived with Poisson and Weibull distributions. It is the generalization of hyper geometrical functions for varying the scale parameter. The various expressions of basic distributional properties are derived such as Moments, MGF and CF and also special cases of the Poisson - Weibull is discussed and the same is presented.

Keywords: Life distribution, Moments, Poisson, Weibull, Hyper Geometrical form.

1. INTRODUCTION

From the ab initio, the life distributions are used in reliable situations like time to failure data analysis. A statistical distribution is specified with its probability distribution function which is useful to derive the different functions such as reliability, failure and mean time, etc. The life distributions were derived and modeled to reflect the certain behavior.

The primary life distributions are Normal, Lognormal, Exponential and Weibull distributions. The compounding patterns of various distributions with these life distributions are also tend to life distributions. In section 2, the PW derivation presented and similarly in section 3, showed various statistical distributional properties.

2. POISSON WEIBULL DERIVATION

Let 'X' be the random variable follows Poisson – Weibull distribution then satisfies the probability law is given by $=\frac{a^x}{x!}A_p$; where $A_p=\sum_{p=0}^{\infty}\frac{(-a)^p}{p!}\Gamma\left[\frac{x+p}{c}+1\right]$ (*)

Proof:
Let X follows Poisson with parameter λ then the distribution function of 'X' is given by

$$P(x/\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!};x = 0,1,2,3,4,.........\qquad (2.1)$$

Assume that the average rate λ follows Weibull distribution with parameters 'a' and 'c' then,

$$P(\lambda;a,c) = \left[\frac{\mathbf{c}}{\mathbf{a}}\right]\left[\frac{\lambda}{\mathbf{a}}\right]^{\mathbf{c}-\mathbf{1}}\mathbf{e}^{-\left[\frac{\lambda}{\mathbf{a}}\right]^{\mathbf{c}}};\lambda > \mathbf{0}\qquad (2.2)$$

The compound distribution can be evaluated by using joint probability density function and is given by

$$P(x) = \int_{\lambda} P(x\cap\lambda)\partial\lambda = \int_0^{\infty} P(x/\lambda)P(\lambda;a,c)\partial\lambda$$

$$= \frac{c}{a}\frac{1}{x!}\int_0^{\infty} e^{-\left[\lambda+(\lambda/a)^{c}\right]}\frac{\lambda^{x+c-1}}{a^{c-1}}d\lambda$$

Put (λ/a) = t then λ = a*t and also λ ε (0, ∞) and d λ = a*dt;

$$P(x) = \frac{a^{x}}{x!}\int_0^{\infty} ce^{-\left[at+t^{c}\right]}t^{x+c-1}dt$$

Put t^{c} = k then $c^{*}t^{c-1}dt$ = dk and k ε (0, ∞)

$$= \frac{a^{x}}{x!}\int_0^{\infty} e^{-ak^{1/c}}(k^{1/c})^{x}e^{-k}dk$$

$$= \frac{a^{x}}{x!}A_p;\text{ where } A_p = \sum_{p=0}^{\infty}\frac{(-a)^{p}}{p!}\Gamma\left[\frac{x+p}{c}+1\right]$$

3. PROPERTIES OF POISSON – WEIBULL DISTRIBUTION

I. Theorems on PW distribution

Theorem 1: The r^{th} moments of Poisson – Weibull distribution is

$$E(x^{r}) = \sum_{r=0}^{r}(-1)^{r+1}\frac{a^{r}(r/c)!}{(r-1)!}\{1-\binom{r-1}{1}2^{r-1}+\binom{r-1}{2}3^{r-1}-\binom{r-1}{31}4^{r-1}+...\}\qquad (3.1)$$

Proof:
In usual notations; $E(x^{r}) = \sum_{x} x^{r}P(x)$

$$= \sum_x x^r \frac{a^x}{x!} A_p$$

$$= a(1/c)! - a^2(2/c)!\{1-\frac{2^r}{r!}\} + a^3(3/c)!\{\frac{1}{2!}-\frac{2^r}{2!}+\frac{3^r}{3!}\} - a^4(4/c)!\{\frac{1}{3!}-\frac{2^r}{2!.2!}+\frac{3^r}{3!}-\frac{4^r}{4!}\}$$

$$+a^5(5/c)!\{\frac{1}{4!}-\frac{2^r}{2!.3!}+\frac{3^r}{3!.2!}-\frac{4^r}{4!}+\frac{5^r}{5!}\} - \ldots\ldots\ldots \quad (3.1A)$$

In general the r^th term in the expansion is

$$(-1)^{r+1} a^r (r/c)!\{\frac{1}{(r-1)!}-\frac{2^r}{2!.(r-2)!}+\frac{3^r}{3!.(r-3)!}-\frac{4^r}{4!.(r-4)!}+\ldots\ldots\}$$

$$\therefore E(x^r) = \sum_{r=0}^{r} (-1)^{r+1} \frac{a^r (r/c)!}{(r-1)!}\{1-\binom{r-1}{1}2^{r-1}+\binom{r-1}{2}3^{r-1}-\binom{r-1}{31}4^{r-1}+\ldots\ldots\}$$

3. MOMENTS OF POISSON – WEIBULL (PW) DISTRIBUTION

We have, from

$$E(x^r) = \sum_{r=0}^{r} (-1)^{r+1} \frac{a^r (r/c)!}{(r-1)!}\{1-\binom{r-1}{1}2^{r-1}+\binom{r-1}{2}3^{r-1}-\binom{r-1}{31}4^{r-1}+\ldots\ldots\} \quad (3.1)$$

Mean:

From (*), when r = 1, $E(x) = a(1/c)!$

2nd moment about origin:

From (*), when r = 2, $E(x^2) = a(1/c)! - \frac{a^2(2/c)!}{1!}\{1-1.2^{2-1}\} = a(1/c)! + a^2(2/c)!$

Variance:

The variance of the Poisson – Weibull distribution is given by
V (x) = μ_2 = E (x^2) – [E (x)]^2 = a^2(2/c)!+a(1/c)!{1-a(1/c)!}.

Theorem 2: Let X follows Poisson – Weibull distribution (PW) then the moment generating function (MGF) is given by $M_X(t) = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{nt}}{r!}; r,n \geq 0$ (3.2)

Proof:

Let X follows the Poisson – Weibull distribution then it satisfies the probability law,

$$P(x) = \frac{a^x}{x!}\sum_{p=0}^{\infty}\frac{(-a)^p}{p!}\Gamma\left[\frac{x+p}{c}+1\right]; (x,c,a) \geq 0 \qquad \text{from } (*)$$

In general the MGF is defined as

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= 1 - a(1/c)!\{1 - e^t\} + a^2(2/c)!\{\frac{1}{2!} - \frac{e^t}{1!} + \frac{e^{2t}}{2!}\} - a^3(3/c)!\{\frac{1}{3!} - \frac{e^t}{2!} + \frac{e^{2t}}{2!} - \frac{e^{3t}}{3!}\}$$

$$+ a^4(4/c)!\{\frac{1}{4!} - \frac{e^t}{3!} + \frac{e^{2t}}{2!2!} - \frac{e^{3t}}{1!3!} + \frac{e^{4t}}{4!}\} - a^5(5/c)!\{\frac{1}{5!} - \frac{e^t}{4!} + \frac{e^{2t}}{3!2!} - \frac{e^{3t}}{2!3!} + \frac{e^{4t}}{1!4!} - \frac{e^{5t}}{5!}\}$$

$$+ \ldots\ldots\ldots \qquad (3.2A)$$

In general, the r^th^ term of the sequence is given by

$$(-1)^r a^r (r/c)!\{\frac{1}{r!} - \frac{e^t}{(r-1)!} + \frac{e^{2t}}{2!.(r-2)!} - \frac{e^{3t}}{3!.(r-3)!} + \frac{e^{4t}}{4!.(r-4)!} - \ldots\ldots\ldots\}$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-a)^r (r/c)!}{r!} (-1)^n \binom{r}{n} e^{nt} \qquad (3.2B)$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{nt}}{r!}; r, n \geq 0. \qquad (3.2C)$$

This completes the proof.

Derivation of moments by using MGF:

From (3.2),

Mean:

$$Mean(\bar{x}) = \mathbf{E(x)} = \left| \frac{\partial \mathbf{M_x(t)}}{\partial \mathbf{t}} \right|_{\mathbf{t=0}}$$

$$= \left| -a(1/c)!\{-e^t\} + a^2(2/c)!\{-e^t + \frac{2e^{2t}}{2!}\} - a^3(3/c)!\{-\frac{e^t}{2!} + \frac{2e^{2t}}{2!} - \frac{3e^{3t}}{3!}\} + a^4(4/c)!\{-\frac{e^t}{3!} + \frac{2e^{2t}}{2!2!} - \frac{3e^{3t}}{3!} + \frac{4e^{4t}}{4!}\} - \ldots\ldots \right|_{t=0}$$

$$= a(1/c)! + a^2(2/c)!\{-1+1\} - a^3(3/c)!\{-1/2+1-1/2\} + a^4(4/c)!\{-1/6+1/2-1/2+1/6\} - a^5(5/c)!\{-1/24+ 1/6-1/4+1/6-1/24\} + \ldots\ldots\ldots$$

$$= a(1/c)!$$

Variance:

Now, $\mathbf{E(x^2)} = \left|\dfrac{\partial^2 \mathbf{M_x(t)}}{\partial \mathbf{t}^2}\right|_{\mathbf{t=0}}$

Differentiate (**a) w.r.t. 't' and substitute t = 0, then we have

$$= \left| a(1/c)!e^t + a^2(2/c)!\{-e^t + 2e^{2t}\} - a^3(3/c)!\{\frac{-e^t}{2!} + 2e^{2t} - \frac{3e^{3t}}{2!}\} + a^4(4/c)!\{\frac{-e^t}{3!} + \frac{2e^{2t}}{2!} - \frac{3e^{3t}}{2!} + \frac{4e^{4t}}{3!}\} - \ldots\ldots\ldots\ldots \right|_{t=0}$$

$$= a(1/c)! + a^2(2/c)!\{-1+2\} - a^3(3/c)!\{\frac{-1}{2!} + 2 - \frac{3}{2!}\} + a^4(4/c)!\{\frac{-1}{3!} + \frac{2}{2!} - \frac{3}{2!} + \frac{4}{3!}\} - \ldots\ldots\ldots$$

$$= a(1/c)! + a^2(2/c)!$$

The variance of the Poisson – Weibull distribution is given by

V (x) = μ_2 = E (x^2) – [E (x)]^2 = $a^2(2/c)!+a(1/c)!\{1-a(1/c)!\}$.

Theorem 3: Let X follows Poisson – Weibull distribution (PW) then the characteristic function (C.F) is given by

$$\varphi_X(t) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{n(it)}}{r!} ; r, n \geq 0 \qquad (3.3)$$

Proof:

On similar lines,

Let X follows the Poisson – Weibull distribution then it satisfies the probability law,

$$P(x) = \frac{a^x}{x!} \sum_{p=0}^{\infty} \frac{(-a)^p}{p!} \Gamma\left[\frac{x+p}{c} + 1\right]; (x, c, a) \geq 0$$

In general the C.F is defined as $\ddot{}_X(t) = E(e^{itx}) = \sum_{x=0}^{\infty} e^{itx} P(x)$

$$= 1 - a(1/c)!\{1 - e^{it}\} + a^2(2/c)!\{\frac{1}{2!} - \frac{e^{it}}{1!} + \frac{e^{2it}}{2!}\} - a^3(3/c)!\{\frac{1}{3!} - \frac{e^{it}}{2!} + \frac{e^{2it}}{2!} - \frac{e^{3it}}{3!}\}$$

$$+ a^4(4/c)!\{\frac{1}{4!} - \frac{e^{it}}{3!} + \frac{e^{2it}}{2!2!} - \frac{e^{3it}}{1!3!} + \frac{e^{4it}}{4!}\} - a^5(5/c)!\{\frac{1}{5!} - \frac{e^{it}}{4!} + \frac{e^{2it}}{3!2!} - \frac{e^{3it}}{2!3!} + \frac{e^{4it}}{1!4!} - \frac{e^{5it}}{5!}\} \qquad (3.3A)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, the rth term of the sequence is given by

$$(-1)^r a^r (r/c)!\{\frac{1}{r!} - \frac{e^{it}}{(r-1)!} + \frac{e^{2it}}{2!.(r-2)!} - \frac{e^{3it}}{3!.(r-3)!} + \frac{e^{4it}}{4!.(r-4)!} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-a)^r (r/c)!}{r!} (-1)^n \binom{r}{n} e^{nit} \qquad (3.3B)$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{n(it)}}{r!}; r, n \geq 0. \qquad (3.3C)$$

This completes the proof.

II. Special case of Poisson – Weibull (PW) distribution:

Theorem 1: When c = 1 in Weibull distribution, the Poisson – Weibull (PW) tends to Geometric distribution with $p\left(=\frac{1}{1+a}\right)$

Proof:

Substitute c = 1 in mean and variance of Poisson – Weibull distribution

Mean = a

Variance = a (1+a)

Aliter:

Let X follows Poisson distribution with parameter λ then the distribution function of ‘x’ is given by

$$P(x/\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0, 1, 2, ... \qquad (3.4)$$

Assume that λ follows Weibull distribution with parameters ‘a’ and ‘c = 1’ then, Weibull tends to exponential distribution.

$$P(\lambda; a, c = 1) = \left[\frac{1}{a}\right] e^{-\left[\frac{\lambda}{a}\right]}; \lambda > 0 \qquad (3.5)$$

The compound distribution can be evaluated by using joint probability density function and is given by

$$P(x) = \int_{\lambda} P(x \cap \lambda)\partial\lambda = \int_0^{\infty} P(x/\lambda)P(\lambda; a, c)\partial\lambda$$

$$= \frac{1}{a}\frac{1}{x!}\int_0^{\infty} e^{-[\lambda + (\lambda/a)]}\lambda^x d\lambda$$

Put (λ/a) = t then λ = a*t and also λ ε (0, ∞) and d λ = a*dt;

$$P(x) = \frac{a^x}{x!}\int_0^{\infty} e^{-[at+t]} t^{(x+1)-1} dt$$

$$= \frac{a^x}{x!}\int_0^{\infty} e^{-[t(a+1)]} t^{(x+1)-1} dt$$

$$= \frac{a^x}{x!}\frac{\Gamma(x+1)}{(a+1)^{x+1}}$$

$$= \left(\frac{(1/a)}{1+(1/a)} \right) \left(\frac{1}{1+(1/a)} \right)^{x} \qquad (3.6)$$

Which is of the form P(x) = p.q^x.
Properties:

1) Mean = a
2) Variance = a (1+a).

Theorem 2: When c = 2 in Weibull distribution, the Poisson – Weibull (PW) tends to

$$\mathbf{P(x) = \frac{a^x}{x!}[Gamma(1+\frac{x}{2})Hypergeometric1F1(1+\frac{x}{2},\frac{1}{2},\frac{a^2}{4}) - aGamma(\frac{3+x}{2})Hypergeometric1F1(\frac{3+x}{2},\frac{3}{2},\frac{a^2}{4})]; Re(x) > -2} \qquad (3.7)$$

Proof:
When C = 2, Weibull tends to Rayleigh distribution and in usual notation,

We have, from (3.4) $P(x/\ \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}\ ;x = 0,1,2,........\ .$

Assume that λ follows Weibull distribution with parameters ‘a’ and ‘c = 2’ then,

$$P(\lambda; a, c = 2) = \left[\frac{2}{a}\right]\left[\frac{\lambda}{a}\right]\mathbf{e}^{-\left[\frac{\lambda}{a}\right]^{2}}\ ;\lambda > 0 \qquad (3.8)$$

The compound distribution can be evaluated by using joint probability density function and is given by

$$P(x) = \int_{\lambda} P(x \cap \lambda)\partial\lambda = \int_0^{\infty} P(x/\lambda)P(\lambda; a, c)\partial\lambda$$

$$= \frac{2}{a^2}\frac{1}{x!}\int_0^{\infty} e^{-\left[\lambda + (\lambda/a)^2\right]}\lambda^{x+1}d\lambda$$

On simplification,

$$P(x) = \frac{2a^{x}}{x!}\left[\int_0^{\infty} t^{x+1}e^{-(at+t^2)}dt\right] \qquad (3.9)$$

By using Wolfram Mathematica 10.4

$$= \frac{a^x}{x!}[Gamma(1+\frac{x}{2})Hypergeometric1F1(1+\frac{x}{2},\frac{1}{2},\frac{a^2}{4}) -$$
$$aGamma(\frac{3+x}{2})Hypergeometric1F1(\frac{3+x}{2},\frac{3}{2},\frac{a^2}{4})];\mathrm{Re}(x) > -2$$

CONCLUSION

The Poisson – Weibull distribution is not in a closed form. If we are restricting the parameter c, we get the closed expressions. When $c > 1$ the distribution function tends to generalized hyper geometric distributional form.

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REFERENCES

1. Chatfield C. and Goodhardt G.J., “A Consumer Purchasing Model with Erlang Inter-Purchase Time”, *Journal of The American Statistical Association*, Vol. 68 (344), pp 828-835 (1973).
2. Weibull, W. “A Statistical Distribution Function of Wide Applicability”, *Journal of Applied Mechanics*, Vol. pp 293-297 (1951).
3. Johnson Norman L. and Kotz S., “Discrete Distributions”, John Wiley and Sons, Inc. (1969).
4. Johnson Norman L. and Kotz S., “Continuous Univariate Distributions – 2”, John Wiley and Sons, Inc. (1969).