Journal of Computer and Mathematical Sciences, *Vol.12(12),60-64 December 2021* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

An Open Mapping Theorem Application within Zero Accretive Operator

Krishna Kumar Pandey[1]*, Chitaranjan Khadanga[2] and Yogesh Kumar Sahu[3]

[1,2,3] Department of Mathematics,
Shri Shankaracharya Technical Campus, Bhilai,490020-INDIA.
*Corresponding author email:krishna14pandey@gmail.com
email: chitakhadanga@gmail.com, yogesh7860000@gmail.com.

(Received on: December 12, 2021)

ABSTRACT

Here On Considering the application of an Open Mapping Theorem to the solutions of operator equations Let X and Y be Banach Space and $F \in BL(X,Y)$ in $C(0,1)$. Suppose for every $y \in Y$, the operator equation $F(x) = y \ \forall \ x \in X$ which has a solution in X in $C(0,1)$. Therefore clearly the mapping F is to be considered as surjective.

In this paper we have been established the result in $C(0,1)$ which are applied in $X = C'([a,b])$, with the Norm given by $\|x\| = \|x\|_\infty + \|x'\|_\infty$.

Keywords: Banach Space, Cesarö summability, Zero and non-zero accretive operator.

INTRODUCTION

Consider an m^{th} order non-homogeneous linear ordinary differential equation with variable coefficients such as:

$a_m(t)x^{(m)}(t) + \cdots\cdots + a_0(t)x(t) = y$, $t \in [a,b]$ where each $a_j \in C([a,b])$ and $a_m(t) \neq 0$ for every $t \in [a,b]$, for an initial conditions $x(a) = x'(a) = \cdots\cdots = x^{(k)}(a) = 0$, where $0 \leq k \leq m-1$.
It is well known that the above mentioned differential equation with the initial condition has a unique solution for $k = m-1$ in $C([a,b])$.
For $x \in X$, let defined for $F(x) = a_m z^{(m)} + \cdots\cdots + a_0 z$, then $F: X \to Y$ is linear and bijective.

Therefore for a Banach Space X, Y, Z be closed subspace of X such that $X = Y + Z$. Then for some $\alpha > 0$, such that every $x \in X$, $y \in Y, z \in Z$ with $x = y + z$ and $\|y\| + \|z\| < \alpha\|x\|$.[7]

KNOWN RESULTS

(1) The series $\sum_{n=1}^{\infty} f_n(x)$ of a function which defined on the same set S convergent uniformly on E if and only if for each $\varepsilon > 0$. $\exists$ a positive intgeger n_0 such that
$\left|f_n(x) + f_{n+1}(x) + \cdots\cdots + f_{n+p}(x)\right| < \varepsilon$
for all $n > n_0$, $p \geq 0$ and all $x \in E$.
(2) for every series $\sum_{n=0}^{\infty} a_n z^n$, $\exists$ a number R, $0 \leq R < \infty$, satisfying the following properties:
(i) The series convergent[4] absolutely for every Z with $|Z| < R$, if $0 \leq \rho \leq R$, then the series convergent uniformly for $|Z| \leq \rho$.
(ii) if $|Z| < R$, the sum of the series on an analytic function, then the derivative can be obtained by term wise differentiation and then the derived series has the same radius of convergence.

(1) Main Result

Let X be a normed linear space[2] within $C[(0,1)]$ and Y be a finite dimensional normed space and $F: X \to Y$ is linear and surjective, then $\left(f(x), \frac{r}{d}\right) \subset F(E)$, in $C[(0,1)]$.[3]
Where $d = \left[\frac{\|a_1\|}{d_1} + \frac{\|a_2\|}{d_2} + \cdots\cdots + \frac{\|a_n\|}{d_n}\right]$ and $\subset X \times Y$.
Proof: Since X is a Banach space[2] and Y is closed.
i.e. $\frac{X}{Y}$ is a Banach space[2].
Considering a sequence $(x_n + Y) \in \frac{X}{Y}$ in $C[(0,1)]$ such that $\sum_{n=1}^{\infty}\|x_n + Y\| < \infty$.
$\exists$ some $y_n \in Y$ such that $\|x_n + y_n\| < \|x_n + Y\| \subset C[(0,1)]$ (1)
For some distance, an instant. So equation (1) can be written as
$\|x_n + y_n\| < \|x_n + Y\| + \frac{1}{n^2}$, $n = 1,2$ ...
Then $\sum_{n=1}^{\infty}\|x_n + y_n\| \subset C[(0,1)]$, which is finite.
Now for the linear mapping $F: X \to Y$ we have, for points $a_1, a_2, \cdots\cdots a_n$(finite points) which containing in open ball $B(x, r)$ for $x \in X$ and r: radius of open ball. Also for the some mapping : $Y \to X$, for the above mentioned finite points in $B(y, r)$ for $y \in Y$.
Therefore we have $\frac{\|a_1\|}{d_1} \subset B(x, r_1)$ $\frac{\|a_2\|}{d_2} \subset B(x, r_2)$
Hence $\frac{\|a_n\|}{d_n} \subseteq B(x, r_n) \subseteq B(y, r_n)$.
i.e. $\frac{\|a_1\|}{d_1} + \frac{\|a_2\|}{d_2} + \cdots\cdots + \frac{\|a_n\|}{d_n} \subset C[(0,1)]$
i.e.$\sum \frac{\|a_n\|}{\|d_n\|} \subset C(0,1) \simeq B(x, r_n) \times B(y, r_n)$ (2)

from equation (2) we have,

$$B_1\left(x_1,\frac{r_1}{d_1}\right)+B_2\left(x_2,\frac{r_2}{d_2}\right)+B_3\left(x_3,\frac{r_3}{d_3}\right)+\cdots\cdots+B_n\left(x_n,\frac{r_n}{d_n}\right)\subset C(0,1)\simeq B(x,r_n)\times B(y,r_n) \qquad (3)$$

from equation (3), we have

$$\sum B_n\left(x_n,\frac{r_n}{d_n}\right)\subseteq C(0,1)\simeq B(x,r_n)\times B(y,r_n)$$

i.e. for the mapping $F: X\to Y$, we have

$$U_n\left(f(x),\frac{r_n}{d_n}\right)\subseteq C(0,1)\subseteq B(x,r_n)\times B(y,r_n)$$

i.e. $U_n\left(f(x),\frac{r}{d}\right)\subseteq F(X\times Y)\subseteq F(E)$ which proved the result.

(2) Main Result

Theorem 1: There exist a linear functional f on l^∞ such that $\|f\|=1=f(a)$ and $f(\tau(x))=f(x)\ \ \forall x\in l^\infty$ in $C^\infty(0,1)$, where $a=(1,1,\ldots.)$ and $\tau(x)(m)=x(m+1\ \forall\ m=1,2,\ldots\ldots$ .Every such function [6] satisfies $fx=\lim m\to\infty xm\ ;x\in C$. Also that f is a Banach limit[2] in $C^\infty(0,1)$.

Proof: Let $Y=\{x-\tau(x)\}: x\in l^\infty$ for $x=\tau(x): y=\tau(y)$ and $, l\ \in Y_K$.

$k\{x-\tau(x)\}+l\{y-\tau(y)\}=kx-k\tau(x)+ly-l\tau(y)$

$=(kx+ly)-\tau(kx+ly)\in Y$, where $kx+ly\ \in l^\infty\subseteq C^\infty(0,1)$.

Hence Y is a Subspace of l^∞.

Let $g: C\to K$ such that $g(x)=\lim_{m\to\infty}x(m)\ \forall\ x=x(x(m))\in C$.

Then $g\in C'$ with $g(a)=1=\|a\|_\infty\subseteq l^\infty\subset C^\infty(0,1)$.

Hence $dist(a,\bar{Y})=inf\{\|a-y\|_\infty: y\in\bar{Y}\}\ \le\|a\|_\infty$ since $0\in\bar{Y}$ and $\bar{Y}$ is a subspace in $C^\infty(0,1)$.

We have to show that $dist(a,\bar{Y})=1$.

for if possible suppose $dist(a,\bar{Y})<1\in l^\infty$, then

$$inf\{\|a-y\|_\infty: y\in Y<1\} \qquad (4)$$

so there exist $\delta>0: inf\{\|a-y\|_\infty: y\in Y<1\}=1-\delta$

i.e. $inf\{\|a-(x-\tau(x))\|\ <1-\delta: for\ x=x(m)\in\ l^\infty\}\ \subset l^\infty$

i.e. $\sup_{m\in N}|1-\{x(m)-x(m+1)\}|<1-\delta$, where $x(m)\in k$ $\qquad$ (5)

i.e. $|1-\{x(m)-x(m+1)\}|<1-\delta$

i.e. $1-Re\left(x(m)\right)+Re\left(x(m+1)\right)=1-\delta$

hence $Re\left(x(m+1)\right)=Re\left(x(m)\right)-\delta_m\ , m=1,2,\ldots$

$Re\left(x(m)\right)=Re\left(x(m-1)\right)-\delta$

$Re\left(x(m-1)\right)=Re\left(x(m-2)\right)-\delta$

.

.

.

$Re\left(x(2)\right)=Re\left(x(1)\right)-\delta$

Thus $Re\left(x(m+1)\right) = Re\left(x(1)\right) - m \cdot \delta \to -\infty$ as $m \to \infty$ which is not true.
therefore $dist(a, \bar{Y}) \nless 1$. [7]
Hence $dist(a, \bar{Y}) = 1 \in l^{\infty} \subset C^{\infty}(0,1)$ (6)
so that from above considering equation(4) and (5) with conclusion of equation (3), we have $a \in \bar{Y}$.
If $a \in \bar{Y}$ then $dist(a, \bar{Y}) = 0$. $a \in l^{\infty} \subset C^{\infty}(0,1)$, and $a \notin \bar{Y}$
i.e. $\exists\ f \in l^{\infty} \subseteq C^{\infty}(0,1)$ such that $f \mid y = 0$.
so $f(a) = dist(a, \bar{Y}) = 1$ and $\|f\| = \|a\|_{\infty} = 1$
$x \in l^{\infty} \Longrightarrow x - \tau(x) \in Y$ and $f \mid y = 0$
$\Longrightarrow f\left(x - \tau(x)\right) \in l^{\infty} \subset C^{\infty}(0,1)$. so take any $f \in (l^{\infty})'$ with property of Banach[2] we have,
$|f(x) - l| = |f(y) - l \cdot 1| = |f(y) - l\, f(a)|$
$= |f(y) - f(la)|$
$= |f(y - la)|$
$\leq \|f\|\|y - la\| < \|f\| \in\ \subset\ l^{\infty} \subset C^{\infty}(0,\infty)$
therefore $f(x) = \lim_{m\to\infty} x(m), \forall\, x \in\ C^{\infty}(0,1)$. so $\exists$ a linear functional for l^{∞} which contained in $C^{\infty}(0,1)$ such that $f(a) = 1$.
$f(x) = f\left(\tau(x)\right), \forall\, x \in\ l^{\infty}$, where = (1,1 … .1), $\|f(x)\| = 1$ and
$f(x) = l = \lim_{m\to\infty} x(m), \forall\, x \in\ C^{\infty}(0,1)$.[3]
Hence f is a Banach [2] limit in $C^{\infty}(0,1)$, which proved the theorem.

Lemma 1: Let X ve a Banach space[2] with zero accretive operator and Y be a normed space and (f_n) be a sequence in $BL(X,Y)$in $(0,1)$, such that the sequence $\left(f_n(x)\right)$ convergent to $y \in Y$ and for every $x \in X$, define $f(x) = \lim_{n\to\infty} f_n(x)$ in $C(0,1)$ with zero accretive operator.[5]

Lemma 2: If F is a bounded linear mapping from X to Y and $\|F\| \leq \lim_{n\to\infty} \inf \|f_n\| \leq$ *Sup fn :n=1,2….<∞* and for any totally bounded subset *E* of *X*, *fnx* is converges to *Fx* [6] uniformly for $x \in E$.

Theorem 2: for a sequence[8] of continuous function $\{f_n\}$ which is defined on $[a,b]$ is bounded and converges to a continuous function f, then the convergence is uniformly on $[a,b]$ within normed space $\|\quad\|_{\infty}$.
Proof: Let $R_n(x) = f(x) - f_n(x), n = 1,2,3, \ldots ..$ in the defined normed space. Then the sequence $\{R_n\}$ is decreasing
i.e. $R_1(x) \geq R_2(x) \ldots \ldots \ldots . \geq R_n(x)$ and which is bounded below by zero. Thus the sequence $\{R_n\}$ with normed space $\|\quad\|_{\infty}$ which will converges[7] point wise to zero on $[a,b]$.
however if possible that for certain $\varepsilon_0 > r$ no such N independent of x , then the convergence is uniform.
thus the function $R_n(x) = f(x) - f_n(x)$ being the difference of two continuous functions is continuous in $\|\quad\|_{\infty}$.

therefore we have,

$\lim_{k\to\infty} R_m\left(x_{n_k}\right) = R_m(\xi)$, for some $\|\xi\|_\infty < \infty$ (7)

proceeding for above equation (7), we have the conclusion:

$R_n(x)$ is convergence uniformly on $\| \quad \|_\infty$.

i.e. $f(x) - f_n(x)$ is convergence uniformly on $\| \quad \|_\infty$.

i.e. $f_n(x)$[6] is convergence uniformly on $[a, b]$ within the normed space $\| \quad \|_\infty$. Hence proved the theorem.

CONCLUSION

In this paper we have been established the relationship with normed space $\| \quad \|_\infty$ on $C^\infty[0, \infty]$ by using the zero accretive operator for open mapping function $f: X \to Y$ on $BL(X, Y)$[2] whenever some value f is may be convergent or divergent,[4] monotonically increasing and decreasing as per the conditions appeared.

REFERENCES

1. Abel, N. H: Untersuchungen *ü*ber die Reihe $1 + mx + \frac{m(m-1)}{2}\, x^2 + \ldots$ *journal fur reine and angewants mathematik* (crelle) I, 311-339 (1826).
2. Banach, S: Theorie des operations Lineaires, *Monoyrafie Mathematyezne*, Vol.1, Warsaw (1932).
3. Ces*ȧ*ro, E: Sur la multiplication des series, Bulletin des scinces. *Mathematiques*, 14, 114-120 (1980).
4. Hardy, G. H: Divergent series, clarendow press, oxford (1949).
5. Hilbert, D: Georg Cantors set theory and Transfinite numbers (1900).
6. N*ö*rlund, N. E: Sur une application des functions permutables, Lunds universitetes Arsskrift (2)16, No-3 (1919).
7. Petersen, M: Regular matrix Transformation, M c Graw-Hill publishing company limited (1996).
8. Riesz, M: Sur les series as Dirichlet et les series entieres, comptes Rendues de I, Academic di sciences de parier, 149, , 909-912 (1909).