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Expansion of Zero Accretive Operator in a Closed Graph Mapping

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ABSTRACT

On considering the application on a closed graph theorem to the solutions of operator equation for Banach Space X and Y and letting $f: X \to Y$ be linear mapping ,which is closed and Surjective, then f is an open mapping.

In this paper we have been established the results in the form: for X be a linear space over K, for subsets U , V, and $k \in K$ such that $U \subset V + KU$, then for every $x \in U$, there is a sequence $\{v_n\} \in V$ such that $x - (v_1 + kv_2 + \cdots\cdots$ $+kn-1vn\in knU$, in zero accretive operator.

Keywords: Accretive operator, Linear mapping, Normed linear space, *Banach Space, Metric space..*

INTRODUCTION

Let X and Y be Metric spaces and F be a mapping from X to Y. Then f is continuous if $x_n \to x$ in X i.e. $f(x_n) \to y \in Y$ i.e. $y = f(x)$. Clearly the continuous mapping is closed. Also it can be easily seen that f is closed iff the set $G_r(f) = \{(x, f(x)) \in X \times$ $Y{:}x\in X, y\in Y$ is closed in $X\times Y$. If the closed mapping f is bijective with respective to zero accretive operator; $C(0,1)$, then f^{-1} is also closed in $C(0,1)$[3].

Moreover if X be a linear space over K. Consider subsets U and V of X, and $k \in K$ such that $U \subset V + kU$. Then for every $x \in U$, $\exists$ a sequence $(v_n) \in V$ such that
$x - (v_1 + kv_2 + \cdots\cdots + k^{n-1}v_n) \in k^n U$ [1] where $n = 1,2,3 \cdots\cdots$
Let $X = C'([a, b])$ and $Y = C([a, b])$ both with the Sup norm $\|x\|_\infty$ [2], then Y will be Banach Space since X is a proper close subspace of Y.

In this paper we have been describe the situation where X is not Banach Space with zero accretive operator and Y is Banach Space in $C([0,1])$, then both open mapping and closed Graph theorem fails to do so.

Lemma 1.1: If $(\alpha < \beta) \geq 0$ and $\lambda \geq 1$, then $[N, p_n^\alpha]_\lambda = \left[N, p_n^\beta\right]_\lambda$ provided $p_n > 0$, when $\alpha = 0$.

Proof: We know that if for $\lambda \geq 1$, $(\bar{N}, p_n)$ is regular and $P_{n-1}^* = O(n|p_n|)$ and if the series $\sum_{n=0}^{\infty} a_n$ is summable for (N, p_n) [6] to S and summable $|N, p_n|_\lambda$, then the series is summable $[N, p_n]_\lambda$ to S.

i.e. $[N, p_n]_\lambda \Longrightarrow S$ (1)

and if either

(i)$\beta > \alpha$ for $\alpha > -1$

(ii) $\beta > \alpha = -1, p_n > 0$ and $p_n \to \infty$ then

$(N, p_n^\alpha) \Longrightarrow \left(N, p_n^\beta\right)$ (2)

In case for $\lambda = 1$ and from the result for if $(n, p_n) \Longrightarrow (N, q_n)$ and $\sum_{r=0}^{n} |K_{n-r}||p_r| = O(|q_n|)$,then

$[N, p_n]_\lambda \Longrightarrow [N, q_n]_\lambda$ for $\lambda > 1$. (3)

Because for $\sum_{r=0}^{n} \epsilon_r^{\beta-\alpha-1} p_{n-r}^\alpha = p_n^\beta$ and $\epsilon_r^{\beta-\alpha-1} > 0$ (4)

therefore equation (1) and (2) clearly applicable for equation (3).

Hence from equation (4), we have $[N, p_n^\alpha]_\lambda = \left[N, p_n^\beta\right]_\lambda$, which proved the result.

KNOWN RESULTS

1.1: for a non-empty closed convex subset of a Hilbert space H and $x \in H$, then $\exists$ finite approximation from E to x. In particular there is finite elements in E of minimal norm. where E is nonempty closed convex subset of H.

1.2: If for $f \in BL(X, Y), f \neq 0$ and $\alpha \geq 0$, then $\inf\{\|x\|: x \in X, \|f(x)\| = \alpha\} = \frac{\alpha}{\|f\|}$ and $dist\left(\{a, \{x \in X: f(x) = k\}\}\right) = \frac{|f(a)-k|}{\|f\|}$,in particular if $a \notin z(f)$, then $\|f\| = \frac{|f(a)|}{dist\{a, z(f)\}}$.

1.3: If (N, p_n) is regular, then for $\lambda > 0$, $(C, \alpha) \Longrightarrow (N, p_n^\alpha)$ and $[C, \alpha]_\lambda \Longrightarrow [N, p_n^\alpha]_\lambda$ for $\lambda > 0$. for the regularity of (N, p_n) means $(C, 0) \Longrightarrow (N, p)$.

Since $[C, 0]_\lambda$ is more restrictive assumption then $(C, 0)$ we felt the strongly regular is not appropriate form, but analogous remarks apply to absolutely regular. Now the object of the main theorem in this paper is that to investigate results of strongly cesaro regularity method in normed linear space.

Theorem 1.1: If for $\lambda \geq 1$, (N, p_n) is strongly regular with index λ , then (N, p_n) is regular.

Proof: we have suppose that $\{p_n\}$ is ultimately non-increasing, and that $p_n = O\left(\frac{p_n}{n}\right)$, then there is a series which is summable for $[C, 0]_\lambda$ $\forall\, \lambda \geq 1$. But which is not summable for $[N, p_n]_\lambda$ because of (N, p_n) is regular for $\lambda > 0$ and

$$(C, \alpha) \Longrightarrow (N, p_n^\alpha) \text{ and } [C, \alpha]_\lambda \Longrightarrow [N, p_n^\alpha]_\lambda \text{ for } \lambda > 0 \tag{5}$$

thus equation (5) is equivalent to the assertion that

$$\sum_{r=0}^{n} p_r^{1-\lambda} |p_{r-1} - p_r|^\lambda = O(|p_n|) \tag{6}$$

The equation (6) holds only when for $\lambda = 1$.

i.e. $\sum_{r=0}^{n} |p_r - p_{r-1}| = O(|p_n|)$

hence $p_n = \sum_{r=0}^{n} (p_r - p_{r-1}) \leq \sum_{r=0}^{n} |p_r - p_{r-1}| = O(|p_n|)$

i.e. (N, p_n) is regular [4]. This proved the theorem.

In the following results we will extend the results Norlund summability methods in normed linear space.

Theorem 1.2(Main Result): Let X be a normed space and Y is closed subspace of X and $Y \neq X$ in $(C, 0)$ and $(C, 1)$.let r be a real number such that $0 < r < 1$, then $\exists$ some $x_r \in X$ in $(C, 0)$ and $(C, 1)$ such that $\|x_r\| \subseteq (C, 0)$ and $r \leq dist\,(x_r, y) \leq 1 \subseteq (C, 1)$ [2].

Proof: since we have given $Y \neq X$. Consider $x \in X$ such that $x \notin Y$, as the subspace Y is closed in $(C, 0)$. Also as $r < 1$, $\exists$ some $y_0 \in Y \subset (C, 1)$ such that $\|x - y_0\| \leq dist \frac{(x,y)}{r} \subset (C, 1)$.

Let $x_r = \frac{x - y_0}{\|x - y_0\|}$, then choice of x_r depends on $(C, 0)$ and $(C, 1)$.

Clearly in $(C, 0)$, we have $\|x_r\| \leq 1$ and in $(C, 1)$ we have $\|x_r\| = 1$. And since $0 \in Y$, we see that [7] the $dist(x_r, y) \leq 1$ in $(C, 1)$.also since $y_0 \in Y$ we have

$$dist(x_r, y) \subseteq (C, 0) = \frac{dist(x - y_0, y)}{\|x - y_0\|} \tag{7}$$

$$\text{and, } dist(x_r, y) \subseteq (C, 1) = \frac{dist(x - y_0, y)}{\|x - y_0\|} \tag{8}$$

from equation (7) and (8) we have

$$\frac{dist(x_r, y)}{\|x - y_0\|} = \frac{dist(x - y_0, y)}{\|x - y_0\|} \geq r \tag{9}$$

therefore from equation (7), (8) and (9) we have $\|x_r\| \subseteq (C, 0)$. also $r \leq dist(x_r, y) \leq 1 \subseteq (C, 1)$. This proved the result.

Theorem 1.3: let X and Y be normed linear space with zero accretive operator and $f: X \to Y$ be linear, then f is an open mapping if and only if there exist $\gamma > 0$ in $C(0,1)$; such that for every $y \in Y$, there is some $x \in X$ with $f(x) = y$ and $\|x\| \leq \gamma \|y\|$ in $C(0,1)$. In particular for zero accretive operator if for a linear mapping is open then that is subjective.

Proof: Let f be an open mapping in $C(0,1)$. since $U_x(0,1)$ is open in X, the set $f\big(U_x(0,1)\big)$ is open in $Y \in C(0,1)$ .as $0 = f(0) \in f\big(U_x(0,1)\big) \subseteq C(0,1)$,then there is some $\delta > 0$ such that

$\overline{U}_{\gamma}(0,\delta) \subset f\big(U_x(0,1)\big) \subseteq C(0,1)$ (10)

Let consider for $y \in Y, y \neq 0$ then

$\frac{\delta_y}{\|y\|} \in \overline{U}_{\gamma}(0,\delta) \subseteq C(0,1)$ (11)

Hence there is some $x_1 \in U_x(0,1)$ such that $f(x_1) = \frac{\delta_y}{\|y\|}$. letting $x = \frac{\|y\|\, x_1}{\delta}$, we have see that

$f(x) = y$ and $\|x\| < \frac{\|y\|}{\delta} \in C(0,1)$ (12)

Thus from equation (10) and (11) compiling with equation (12), we have $f(x) = y$ and $\|x\| < \frac{\|y\|}{\delta}$. Thus we can let $\gamma = \frac{1}{\delta}$ in $C(0,1)$.

Conversely: let us assume that for every $y \in Y$, there is some $x \in X$, with $f(x) = y$ and $\|x\| \leq \gamma\|y\|$ for some $\gamma > 0$. Consider for an open set E in X and $x_0 \in E$.

Then $U_x(x_0,\delta) \subset C(0,1) \subset E$ for some $\delta > 0$. let $y \in Y$ with $\|y - f(x_0)\| < \frac{\delta}{\gamma}$

By Hypothesis, there is some $x \in X$ such that

$f(x) = y - f(x_0)$ (13)

and $\|x\| \leq \gamma\|y - f(x_0)\|$ in $C(0,1)$ with zero accretive operator. Thus $y = f(x) + f(x_0) = f(x + x_0) \subseteq C(0,1)$, where $x + x_0 \in U_x(x_0,\delta) \subset C(0,1) \subset E$. Since $\|x\| < \delta$, thus

$U_{\gamma} = \left(f(x_0), \frac{\delta}{\gamma}\right) \subset f(E)$ (14)

Hence $f(E)$ is an open set in Y. Therefore by considering equation (13) and (14), we conclude that f is an open mapping in $C(0,1)$ with zero accretive operator.

Theorem 1.4: For a Hilbert space H, $E \subset H$ be a non-empty closed convex subset of H [5]. Then $\exists$ an unique best approximation from E to x, in particular there is an unique element in E of minimal norm.

Proof: let us consider $E \neq \emptyset$. Closed and convex subset of H, for $x \in H$.

Let $d = dist(x,E) = inf\{\|x - y\| : y \in E\}$ so $\exists$ $(y_n) \in E$ such that

$\lim_{n\to\infty}\|x - y_n\| = \alpha$ (15)

i.e. for $\varepsilon > 0$, $\exists\, n_0 : n > n_0 \Longrightarrow \|x - y_n\| < d + \varepsilon$.

Let $m > n_0$ then $\|x - y_m\| < d + \varepsilon$. Since $(x - y_n)$ and $(x - y_m) \in H$. So therefore by parallelogram law, we have

$2\|x - y_n\|^2 + 2\|x - y_m\|^2 = \|2x - (y_n + y_m)\|^2 + \|y_n - y_m\|^2 = 4\left\|x - \left(\frac{y_n+y_m}{2}\right)\right\|^2 + \|y_n - y_m\|^2$

Since $y_n, y_m \in E$ and E is convex , therefore $\frac{y_n+y_m}{2} \in E$.

Hence $\left\|x - \left(\frac{y_n+y_m}{2}\right)\right\|^2 \geq d^2 \Longrightarrow -\left\|x - \left(\frac{y_n+y_m}{2}\right)\right\|^2 \geq -d^2$

therefore $\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\left\|x - \left(\frac{y_n+y_m}{2}\right)\right\|^2 \leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 < \varepsilon$

for $n, m > n_0 \implies \|y_n - y_m\| < \varepsilon$. Hence (y_n) is a Cauchy's sequence in E, which is closed. i.e. y_n is convergent in E. so $\exists\ y \in E$ such that $\lim_{n\to\infty} y_n = y$. by equation (15),
$\|x - \lim_{n\to\infty} y_n\| = d$.
i.e. $\|x - y\| = d = dist(x, E)$, so y is a best approximation from x to E. We know if E is a convex subset of Hilbert space H and if $x \in H$, then there exists at least one best approximation from E to x. Since E is convex, y is the unique best approximation from E to x. since $0 \in H$,
$d' = dist\ (E, 0) = \inf\{\|0 - y\| : y \in H\} = \|y'\|$ for some $y' \in E$.
$\leq \inf\{\|x - y\| \forall\ y \in E\} = d(x, E) = d$
There exists $y' \in E$ such that $\|y'\| \leq d(x, E)$, which proved the result.

CONCLUSION

In this paper we conclude that the investigate results of $(C, 0)$, $(C, 1)$ summability method in Banach space and to apply the transformation from sequence to sequence and sequence to function. Also we have been derived the results of $(C, 0)$, $(C, 1)$ summability method in the field of normed linear space.

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