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Proper Lucky Number of Mesh and It's Derived Architectures

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ABSTRACT

Let $G(V,E)$ be a graph with vertex set V and edge set E . Let f be a labeling defined in G. Define the sum of neighbourhood of vertex v by $\mathrm{s}(v) = \sum_{u\in N(v)} f(u)$, where $N(v)$ denotes the open neighbourhood of $v \in V$. In this paper, we introduce a new labeling called proper lucky labeling. A labeling f is a proper lucky labeling if $f(u) \neq f(v)$ and $\mathrm{s}(u) \neq s(v)$ for all $(u,v) \in E(G)$. The proper lucky number of G, denoted by $\eta_p(G)$ is the least positive integer k such that G has a proper lucky labeling with $\{1,2,\dots,k\}$ as the set of labels. We determined the proper lucky number of Mesh and its derived architectures.

Keywords: Proper labeling, Lucky labeling, Proper lucky labeling, Proper lucky number, Mesh, Extended mesh and Enhanced mesh.

1. INTRODUCTION

Most graph labelling methods trace their origin to one introduced by Rosa in[7], or one given by Graham and Sloane in 1980. For the past decades many researchers studied several interrelated graph labelling problems. These labelling problem have been studied in various different formulations.

Graph coloring is one of the most studied subjects in graph theory. It is an assignment of labels called colors to the elements of a graph, subject to certain constraints. Karonski, Luczak and Thomason[8] initiated the study of proper labeling. The rule of using colors originates from coloring the countries of a map, where each face is colored exactly once. In its simplest outline, vertex coloring or proper labelling is a way of coloring the vertices of a graph such that no two vertices share the same color. The problem of proper labeling offers numerous variants and established great significance at recent times. Graph coloring is used in various research areas of computer science such as networking, image segmentation, clustering, image capturing and data mining.

In recent years, the lucky labeling of graphs were studied by A. Ahai *et al.*,[2] and S. Akbari *et al.*,[3]. Suppose the vertices of a graph G were labeled arbitrarily by positive integers and let $s(v)$ denote the sum of labels over all neighbours of vertex v. A labeling is *lucky* if the function s is a proper coloring of G, that is, if we have $s(u) \neq s(v)$ if u and v are adjacent vertices. The least positive integer k for which a graph G has a *lucky labeling* from the set $\{1,2,\ldots,k\}$ is the *lucky number* of G, denoted by $\eta(G)$. In this paper we introduce a new labeling called proper lucky labeling and obtained the lower bound for proper lucky number using clique number ω. Also, we have completely determined the proper lucky number for mesh and its derived architectures.

Definition: Let $G(V,E)$ be a graph with vertex set V and edge set E . Let f be a labeling defined in G. Define the sum of neighbourhood of vertex v by $s(v) = \sum_{u\in N(v)} f(u)$, where $N(v)$ denotes the open neighbourhood of $v \in V$. A labeling f is a *proper lucky labeling* if $f(u) \neq f(v)$ and $s(u) \neq s(v)$ for all $(u,v) \in E(G)$. The *proper lucky number* of G, denoted by $\eta_p(G)$ is the least positive integer k such that G has a proper lucky labeling with $\{1,2,\ldots,k\}$ as the set of labels.

Example:

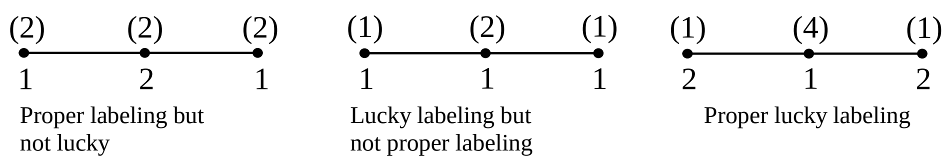

Fig 1. Ilustration of different Labelings

From Fig 1, it is clear that $\eta = 1$ and $\eta_p = 2$.

2. PROPER LUCKY NUMBER AND ITS PROPERTIES

In this section, we obtained the results for proper lucky number connecting with chromatic number χ and clique number ω.

Definition 2.1 [9]: The least number of colors require to color the vertices of a graph so that the adjacent vertices do not have the same color is called the *chromatic number*. It is denoted by χ.

Definition 2.2 [9]: A *clique* C in an undirected graph $G = (V,E)$ is a subset of the vertices $C \subseteq V$, such that any two distinct vertices of C are adjacent. This is equivalent to the condition that the sub graph of G induced by C is complete. A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique. The *clique number* $\omega(G)$ of a graph G is the number of vertices in a *maximum clique* in G.

Theorem 2.1 [1] : The chromatic number of complete graph K_n is $\chi(K_n) = n$.

Theorem 2.2 [2]: The chromatic number of any graph G is greater than or equal to the clique number i.e. $\omega \leq \chi$.

Theorem 2.3 [5]: For any connected graph G, the chromatic number is less than or equal to proper lucky number i.e. $\chi \leq \eta_p$.

Theorem 2.4 : For any connected graph $G,$ let η_p be its proper lucky number and ω be its clique number, then $\omega \leq \eta_p$.

Proof. From the theorems 2.2 and 2.3, we have

$$\omega \leq \chi \leq \eta_p \quad (1)$$

Therefore from (1), we get $\omega \leq \eta_p$.

3. PROPER LUCKY NUMBER OF MESH AND ITS DERIVED ARCHITECTURES

In this section, we have completely determine the proper lucky number of Mesh and its derived architectures.

3.1 Proper Lucky Number of Mesh

Definition: The $m \times n$ mesh denoted $M_{m\times n}$ is defined as the Cartesian product $P_m \times P_n$ of paths on m and n vertices respectively. The number of vertices in $M_{m\times n}$ is mn and its diameter is $m + n - 2$. See Fig 2.

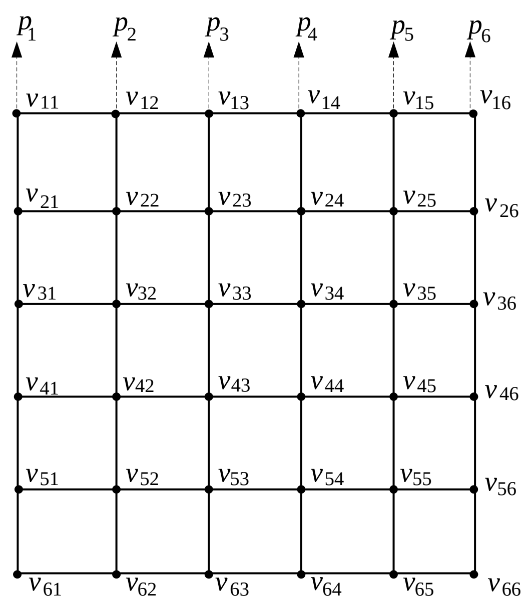

Fig 2. Mesh $M_{6\times 6}$

Theorem 3.1: Let G be a mesh $M_{n\times n}$. Then the proper lucky number of G is $\eta_p(G)$= 2.

Proof. Define a mapping $f\colon V(G) \to \{1,2\}$ as follows, let $v_{ij} \in V$ then the vertex v_{ij} is mapped to 1 or 2 under f such that the adjacent vertices of v_{ij} does not receive that value. i.e. If vertex v_{ij} is mapped to 1, its adjacent vertices is mapped to 2 and vice versa. Clearly, $f(u) \neq f(v)$, for all $(u, v) \in E(G)$. Hence the given labeling is a proper labeling.
Next we claim that the given mapping is a lucky labeling. That is, to prove $s(u) \neq s(v)$ for all $(u, v) \in E(G)$.
We obtain $s(v_{ij})$, the sum of labels over all neighbours of vertex v_{ij} as follows

$s(v_{ij}) = f(v_{ij+1}) + f(v_{i+1\,j})$. $\quad i = 1, j = 1$
$s(v_{ij}) = f(v_{ij+1}) + f(v_{i-1\,j})$. $\quad i = m, j = 1$
$s(v_{ij}) = f(v_{ij-1}) + f(v_{i+1\,j})$. $\quad i = 1, j = n$
$s(v_{ij}) = f(v_{ij-1}) + f(v_{i-1\,j})$. $\quad i = m, j = n$
$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1\,j}) + f(v_{ij+1})$. $\quad i = 2,3 \ldots m-1$ and $j = 1$
$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1\,j}) + f(v_{ij-1})$. $\quad i = 2,3 \ldots m-1$ and $j = n$
$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1\,j}) + f(v_{ij+1}) + f(v_{ij-1})$. $\; i = 2,3 \ldots m-1,\; j = 2,3 \ldots n-1$
$s(v_{ij}) = f(v_{ij-1}) + f(v_{i\,j+1}) + f(v_{i+1j})$. $\quad i = 1$ and $j = 2,3 \ldots n-1$
$s(v_{ij}) = f(v_{ij-1}) + f(v_{i\,j+1}) + f(v_{i-1j})$. $\quad i = m$ and $j = 2,3 \ldots n-1$

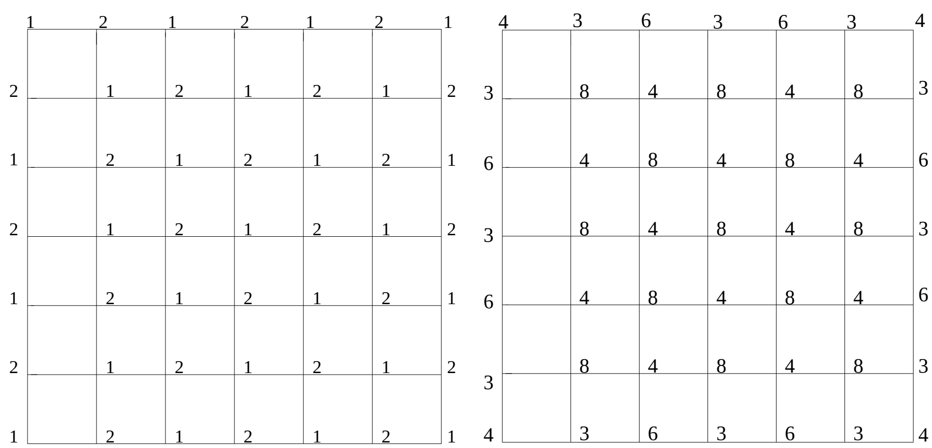

Fig 3. Proper Lucky labeling of Mesh $M_{7\times 7}$ and its sum of neighbourhood

From the above mapping we obtained values for the each neighbourhood of v_{ij}.

Case 1: Inner part of the mesh
$s(v_{2i,2j}) = 8, i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor$.
$s(v_{2i+1,2j+1}) = 8,\; i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor - 1, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor - 1$.

$s(v_{2i,2j+1}) = 4,\ i = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor, j = 1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor - 1.$
$s(v_{2i+1,2j}) = 4,, i = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor - 1, j = 1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor.$

Case 2: Upper part of the mesh
$s(v_{i,2j}) = 3, i = 1,\ j = 1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor.$
$s(v_{i,2j+1}) = 6, i = 1,\ j = 1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor - 1.$
Upper part and Lower part of the mesh have same labeling (when $i = m$).

Case 3: Left side of the grid
$s(v_{2i,j}) = 3,\ i = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor,\ j = 1.$
$s(v_{2i+1,j}) = 6,\ i = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor - 1,\ j = 1.$
Left side and Right side of the mesh have same labeling (when $j = n$).
All the corner vertices receive the same labeling.
From the above cases we see that $s(u) \neq s(v)$ for all $(u, v) \in E(G)$.
Therefore $\eta_p \leq 2$ (1)
Hence the clique number ω of $M_{m\times n}$ is 2, from the theorem 2.4, we have $\eta_p \geq \omega$.
Hence $\eta_p(G) \geq 2$ (2)
Therefore from (1) and (2), we get $\eta_p(G) = 2$.

3.2 Proper Lucky Number of Extended Mesh

Definition: The mesh $M_{m\times n}$ is defined as the Cartesian product $P_m \times P_n$ of paths. The architecture obtained by making each 4-cycle in $M_{m\times n}$ into a complete graph is called an extended mesh. It is denoted by $EX_{m\times n}$. The number of vertices in $EX_{m\times n}$ is mn and its diameter is $min\{m, n\} - 1$. See Fig 4.

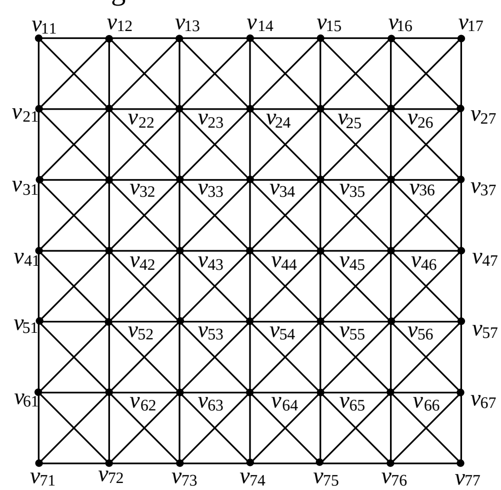

Fig 4. An Extended mesh $EX_{7\times 7}$

Theorem 3.2: Let G be an extended mesh $EX_{m\times n}$. Then the proper lucky number of G is $\eta_p(G)$= 4.

Proof. We Partition the vertex set $V(G_{n\times n})$ into 2 disjoint sets V_1 and V_2.
Let $V_1 = v_{2m(j-1)+i,}\ i = 1,2 \ldots m, j = 1,2 \ldots n$ and
$V_2 = v_{m(2j-1)+i,}\ \ i = 1,2 \ldots m, j = 1,2 \ldots n$.
Define a mapping $f: V(G) \to N$ as follows

$$f\left(v_{2m(j-1)+i}\right) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases} \quad \text{where } i = 1,2 \ldots m, j = 1,2 \ldots \left\lceil \frac{n}{2} \right\rceil.$$

$$f\left(v_{m(2j-1)+i}\right) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases} \quad \text{where } i = 1,2 \ldots m, j = 1,2 \ldots \left\lfloor \frac{n}{2} \right\rfloor.$$

Claim: f is a proper labeling. We need to verify that $f(u) \neq f(v)$, for all $(u, v) \in E(G)$.
Let $e = uw$ be an edge in *G*.

Case 1. Suppose $u, w \in V_1$. Then $u = v_{2m(l-1)+s}$ and $w = v_{2m(l-1)+t}$, $1 \leq l \leq \left\lceil \frac{n}{2} \right\rceil, 1 \leq s, t \leq m$. Since $e = uw$ is an edge, we have $t = s + 1$. Therefore $f(u) = 1, f(w) = 2$ or $f(u) = 2, f(w) = 1$. Hence $f(u) \neq f(v)$.

Case 2. Suppose $u, w \in V_2$. Then $u = v_{m(2l-1)+s}$ and $w = v_{m(2l-1)+t}$, $1 \leq l \leq \left\lceil \frac{n}{2} \right\rceil, 1 \leq s, t \leq m$. Since $e = uw$ is an edge, we have $t = s + 1$. Therefore $f(u) = 3, f(w) = 4$ or $f(u) = 4, f(w) = 3$. Hence $f(u) \neq f(v)$.

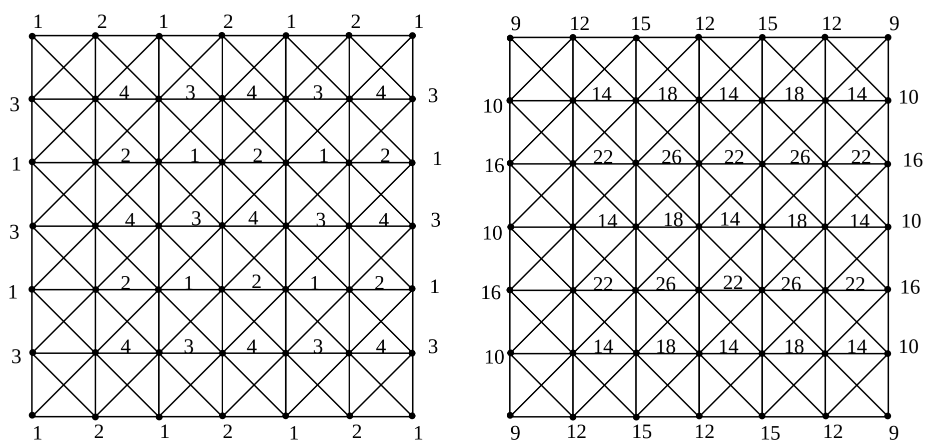

Fig 5. Proper labeling of an Extended mesh $EX_{7\times 7}$ and its sum of neighbourhood

Case 3. Suppose $u \in V_1, w \in V_2$. Then $u = v_{m(2l-1)+s}$ and $w = v_{m(2l-1)+t}$, then the adjacent labelings are shown in fig 5. Since the vertex V_1 is labeled by the 1, 2 and V_2 is labeled by the labels 3, 4. Clearly$f(u) \neq f(v)$, for all $(u, v) \in E(G)$.
Hence the given labeling is a proper labeling.

Next we claim that the given mapping is a proper lucky labeling. That is, to prove $s(u) \neq s(v)$ for all $(u, w) \in E(G)$.
The sum of open neighbourhood of v_{ij} are defined below as follows
When $i = 1, m$ and $j = 1, n$
$s(v_{ij}) = f(v_{ij+1}) + f(v_{i+1\ j}) + f(v_{i+1\ j+1})$.
When $i = 2,3 \ldots\ m-1$ and $j = 1, n$
$s(v_{ij}) = f(v_{i-1\ j}) + f(v_{i+1\ j}) + f(v_{i-1\ j+1}) + f(v_{i\ j+1}) + f(v_{i+1\ j+1})$.
When $i = 2,3 \ldots\ m-1$ and $j = 2,3 \ldots n-1$
$s(v_{ij}) = f(v_{i-1\ j}) + f(v_{i+1\ j}) + f(v_{i+1\ j+1}) + f(v_{i-1\ j+1}) + f(v_{i\ j+1}) + f(v_{i+1\ j-1}) + f(v_{i\ j-1}) + f(v_{i-1\ j-1})$.
When $i = 1, m$ and $j = 2, 3 \ldots n-1$.
$s(v_{ij}) = f(v_{i\ j-1}) + f(v_{i\ j+1}) + f(v_{i+1\ j-1}) + f(v_{i+1\ j}) + f(v_{i+1\ j+1})$.
From the above mapping we obtained values for the each neighbourhood of v_{ij}.

Case 4: Inner part of the Extended mesh
$s(v_{2i,2j}) = 14, i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor$.
$s(v_{2i,2j+1}) = 18,\ i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor - 1$.
$s(v_{2i+1,2j}) = 22,\ i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor - 1, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor$.
$s(v_{2i+1,2j+1}) = 26,, i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor - 1, j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor - 1$.

Case 5: Upper part of the Extended mesh
$s(v_{i,2j}) = 12, i = 1,\ j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor$.
$s(v_{i,2j+1}) = 15, i = 1,\ j = 1,2 \ldots \left\lfloor\frac{n}{2}\right\rfloor - 1$.
Upper part and Lower part of the Extended mesh have same labeling (when $i = m$).

Case 6: Left side of the Extended mesh
$s(v_{2i,j}) = 10,\ i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor,\ j = 1$.
$s(v_{2i+1,j}) = 16,\ i = 1,2 \ldots \left\lfloor\frac{m}{2}\right\rfloor - 1,\ j = 1$.
Left side and Right side of the Extended mesh have same labeling (when $j = n$).
All the corner vertices receive the same labeling.
From the above cases we see that $s(u) \neq s(v)$ for all $uv \in E(G)$.
Therefore $\eta_p \leq 4$ (1)
We note that the maximal complete subgraph of $EX_{m\times n}$ is k_4. Hence the clique number ω of $EX_{m\times n}$ is 4, from the theorem 2.4, we have $\eta_p \geq \omega$.
Hence $\eta_p(G) \geq 4$ (2)

Therefore from (1) and (2), we get $\eta_p(G) = 4$.

3.3 Proper Lucky Number of Enhanced Mesh

Definition: In an $m \times n$ mesh if we place a vertex in each bounded face and join it to the corner vertices of the face, the architecture obtained is called *enhanced mesh*. It is denoted by $EN_{m\times n}$. See Fig 6.

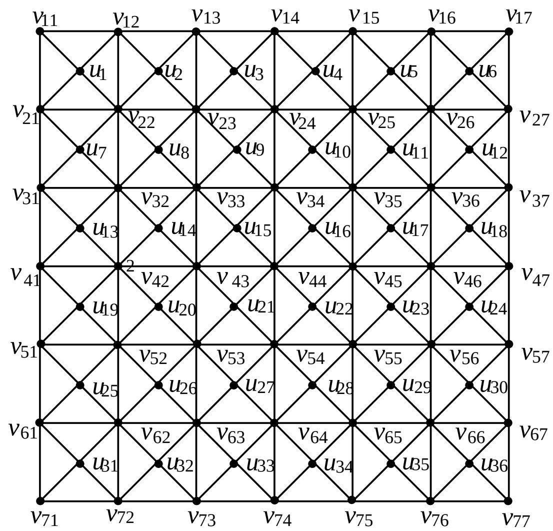

Fig 6. An Enhanced mesh $EN_{7\times 7}$

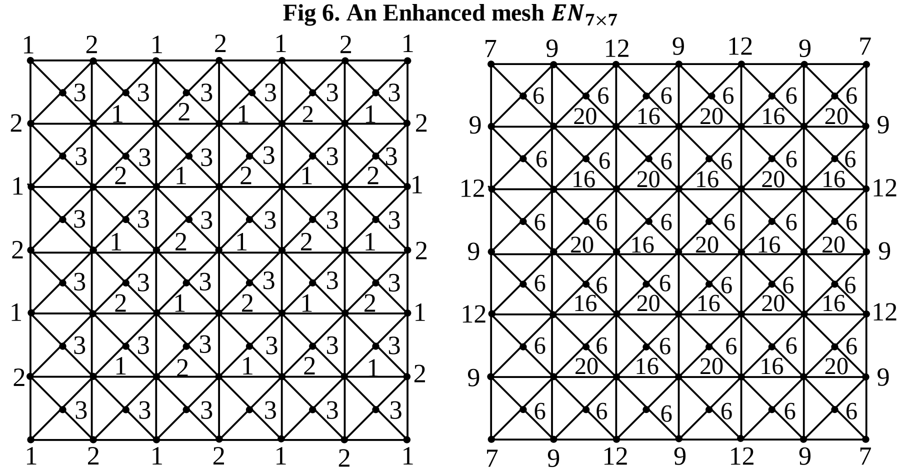

Fig 7. Proper labeling of an Enhanced mesh $EN_{7\times 7}$ and its sum of neighbourhood

Theorem 3.3: The proper lucky number of an enhanced mesh $EN_{m\times n}$ is $\eta_p(EN_{m\times n}) = 3$.

Proof. Define a mapping $f: V(G) \to N$ as follows

$f\left(v_{2i-1,\ j}\right) = \begin{cases} 1, & \text{if } j \text{ is odd} \\ 2, & \text{if } j \text{ is even} \end{cases}$ where $i = 1,2 \ldots \left\lceil \frac{m}{2} \right\rceil, j = 1,2 \ldots n.$

$f\left(v_{2i,\ j}\right) = \begin{cases} 2, & \text{if } j \text{ is odd} \\ 1, & \text{if } j \text{ is even} \end{cases}$ where $i = 1,2 \ldots \left\lceil \frac{m}{2} \right\rceil, j = 1,2 \ldots n.$

$f(u_i) = 3$ where $i = 1,2 \ldots (m-1)(n-1).$

The proof is similar to the theorem 3.2.

CONCLUSION

In this paper, we introduced a new labeling called proper lucky labeling and we obtained the proper lucky number for mesh and its derived architectures. Futher, we investigate the problems in various interconnection networks such as butterfly, benes, honeycomb etc.

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