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Leech Tree Matrix and Leech Tree Energy

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ABSTRACT

Let T be any tree and P be set of paths, given a labelling $w: E(T) \to Z^+$ of the edges of T for a path $p \in P$, such that $w(p) = \sum_{e \in p} w(e)$. Here each path in T has distinct sum weights along its edges and those sums are the consecutive positive integers 1 through $\binom{n}{2}$. Such labelling of a T is a Leech tree labelling. In this article we define a *Leech tree* matrix $\ell(T)$, as follows;

$$\ell(T) = (a_{ij}) = \begin{cases} 1, & if \;\; w(p) \; is \; even \\ -1, & if \;\; w(p) \; is \; odd \\ 0, & otherwise \end{cases}$$

and discuss some spectral properties of *Leech tree* matrix and deduce *Leech tree* energy $E_\ell(T) = E(K_n) = 2(n-1)$.

AMS Subject Classifications: 05C50.

Keywords: *Leech tree*, *Leech tree* matrix, *Leech tree* spectrum and *Leech tree* energy.

INTRODUCTION

A labeled graph is a graph whose vertices or edges or both are labeled by integers with some conditions. If there are no constraints, then every graph can be labeled in infinitely many ways. Graph labelings were first introduced by A. Rosa during 1960's. Since then several graph labeling techniques have been studied.

John Leech in 1975[1], introduced the problem of labelling the edges of a tree with distinct positive integers, so that, the sums along distinct paths in tree were distinct and the set of such path sums were consecutive starting with 1, such a labelled tree is called *Leech tree.*

In otherwards, one can assign positive edge weights to its edges, such that the $\binom{n}{2}$ path weights, i.e. the sums of weights along the $\binom{n}{2}$ distinct paths connecting the pairs of the n vertices of the tree, yield exactly the numbers $1, 2, 3, \dots, \binom{n}{2}$. Since edges of the tree are also paths, the edge weights have to be positive integers as well.

The motivation there was an application in electric engineering such a labelling would give a way of constructing a universal resistor with possible impedance of 1 upto $\binom{n}{2}$ units from $n-1$ simple resistors. The difficulty of the existence problem lies in the unusual way of mixing additive number theory with combinatorics. One can also see several hard problems viz., The graceful tree conjecture of Ringel[2] and Entringers prime labelling conjecture[3].

Herbert Taylor[4] gave a beautiful proof restricting the number of vertices on which *Leech tree* can lie, interested reader can also see[5].

Theorem 1:[4]. If there is *Leech tree* on n vertices, then $n = k^2$ or $n = k^2 + 2$.

Till today, the only known *Leech tree* of order $n \leq 6$, are listed in figure 1, below. This leads to the conjecture, there are finitely many *Leech trees*.

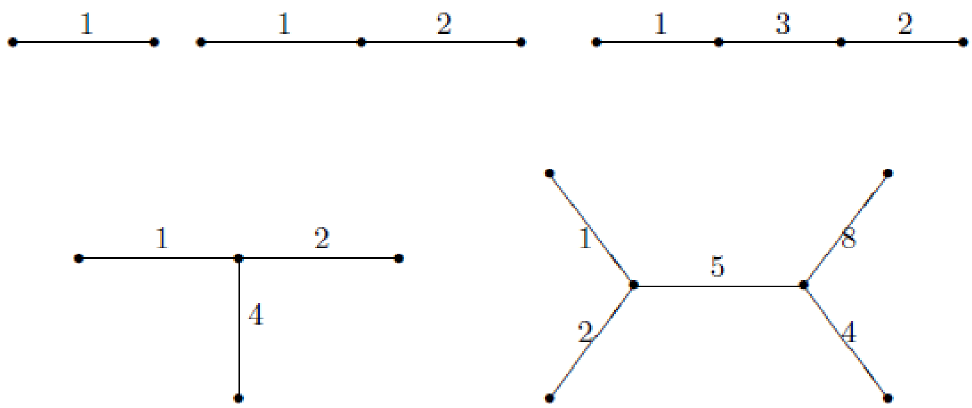

Figure 1: The known Leech trees.

More recently, Szekely, Wang and Zhang[6,7] performed a computer search to show the non-existence of Leech trees of order 9 and 11. They also provided upper bound on the diameter and maximum degree.

Theorem 2: [6]. If there is a *Leech tree* on n vertices, then it has no paths longer than $\frac{n}{\sqrt{2}}(1 + o(1))$.

Theorem 3: [6]. In Leech tree, the maximum degree of a vertex is at most

$$d \leq \left(\frac{\sqrt{8}}{3} + o(1)\right) n$$

Now define *Leech tree* matrix as follows;

$$\ell(T) = \left(a_{ij}\right) = \begin{cases} 1, & if \;\; w(p)\ is\ even \\ -1, & if \;\; w(p)\ is\ odd \\ 0, & otherwise \end{cases}$$

Thus the, matrix $\ell(T)$ is real symmetric matrix with entries $(-1, 0, 1)$. The characteristic polynomial of $\ell(T)$ is $P(\ell(T), \lambda) = a_0\lambda^{\mathrm{n}} + a_1\lambda^{\mathrm{n}-1} + \cdots + a_n$. The eigenvalues of $\ell(T)$ are the roots of characteristic polynomial of $\ell(T)$, thus the spectrum of $\ell(T)$ is

$$Spec\left(\ell(T)\right) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots\ldots. & \lambda_{\mathrm{r}} \\ \mathrm{m}_1 & \mathrm{m}_2 & & \mathrm{m}_{\mathrm{r}} \end{pmatrix}, r \leq n$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{r}}$ are eigenvalues of *Leech tree* with their multiplications, $\mathrm{m}_1, \mathrm{m}_2, \ldots, \mathrm{m}_{\mathrm{r}}$, The concept of energy has been much intensified in the last two decades see [7], and references therein. Now we formally define a *Leech tree* energy. The *Leech tree* energy is, the absolute sum of eigenvalues of *Leech tree* matrix $\ell(T)$, i.e., $E_\ell(T) = \sum_{i=1}^{n} |\lambda_{\mathrm{i}}|$.

Suppose that, $P(\ell(T),\ \lambda) = \det\left(\lambda I - \ell(T)\right) = a_0\lambda^{\mathrm{n}} + a_1\lambda^{\mathrm{n}-1} + \cdots + a_n$ is the characteristic polynomial, of $\ell(T)$, then the coefficient c_i can be interpreted as principal minors of $\ell(T)$ and this leads to the following results.

Theorem 5. Using above notation, we have

i. $c_0 = 1$

ii. $c_1 = 0$

iii. $c_2 = -\sum e(p')\,, where\ p'\ is\ the\ largest\ \ path\ in\ T\ with\ maximum\ wieghts.$

iv. $c_n = -m$ (number of edges in G).

Proof: i). It follows from the definition $P(\ell(T), \lambda) = \det\left(\lambda I - \ell(T)\right)$, that $c_0 = 1$, for each $i \in \{1,2, \ldots. n\}$, the number $(-1)c_i$ is the sum of those principal minor of $\ell(T)$, which have i rows and columns.

ii). Since the graph is simple, all the diagonal entries must be zero, thus $c_1 = 0$.

iii). $c_2 = (-1)^{n-2} \times Sum\ of\ the\ principal\ minors\ of\ order\ 2$

$$= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1 \leq i < j \leq n} \left(a_{ii}a_{jj} - a_{ij}a_{ji}\right) = -\sum_{1 \leq i < j \leq n} {a_{ij}}^2 = -\sum e(p')$$

iv). Lastly $c_n = \det\left(\ell(T)\right) = -m$ (number of edges in G).

Theorem 6. If $\lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_{\mathrm{n}}$ are the eigenvalues of $\ell(T)$, then

i. $\sum \lambda_i = 0$

ii. $\sum {\lambda_i}^2 = 2m + m(m-1)$.

Proof. i). Since the diagonal elements are zero that equals the trace of $\ell(T)$. Hence the result.

ii). Consider $\sum {\lambda_i}^2 = \sum_{i<j} \sum_{i<j} a_{ij}\, a_{ji} = 2\sum_{i<j} \left(a_{ij}\right)^2 + \sum_{i=1} (a_{ii})^2$

$= 2\sum_{i<j} \left(a_{ij}\right)^2 = 2m + m(m-1)$.

Theorem 7. For any *Leech tree* with n vertices $E_\ell(T) = E(K_n) = 2(n-1)$.

Proof. By direct computations of characteristic polynomial of $\ell(T)$, we noticed that characteristic polynomial and spectrum of $\ell(T)$ share the characteristic polynomial and spectrum of adjacency matrix of K_n respectively, thus the;
$Spec(\ell(T)) = \begin{pmatrix} (n-1), & -1 \\ 1, & (n-1) \end{pmatrix}$, thus $E_\ell(T) = 2(n-1) = E(K_n)$.

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