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Generating Function for - $\overline{r - Ga_i\, spt_i(n)}^{\,j}$

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ABSTRACT

In this paper we obtain the generating function for the number ith smallest parts of r-jth over Ga partition of n in which parts are distinct and ith smallest part is the first part.

Keywords: generating function, number ith smallest parts of r-jth over Ga partition of n.

INTRODUCTION

The Corteal and Love Joy[1] initiated the study of overpartitions jth overpartitions were introduced by Ramabhadra Sarma I and Hanuma Reddy K and a formula for the number of ith smallest parts for overpartitions was obtained by Hanuma Reddy and Manjusri[2]. Ga partitions were introduced by Hanuma Reddy.K and Sager G.V.R.K[3]. For more details on theory of partitions one can refer[4,5,6].

1.1 Definitions: A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$ and is denoted by $n = (\lambda_1, \lambda_2, ..., \lambda_r)$,
$n = \lambda_1 + \lambda_2 + \lambda_3 + ...\lambda_r$.

Each λ_i is called a part of λ. We also write $\lambda = (\mu_1^{f_1}, \mu_2^{f_2} .....\mu_s^{f_s})$
where μ_i^s are the distinct parts of λ_i^s and f_i is the number of times of occurence of μ_i in the partition λ.

Overpartitions: A j^{th} overpartition of n is a partition of n in which first (equivalently, the final) occurrence of a number is overlined successively up to j times.

- **A** $Ga\ partition$ of n is a partition of n in which the smallest part is of the form a^{k-1} for some $k \in N$
- **A** $r-Ga_i\ partition$ of n is an r-partition of n with i^{th} smallest parts of the form a^{k-1}
- $r-Ga_i spt_j(n)$ denotes the j^{th} smallest part of a *r-* $Ga_i\ partition$ of n $Ga_i spt_j(n)$ =sum of j^{th} smallest parts of Ga_i partitions of n
- A $r-j^{th} overpartition$ whose i^{th} smallest parts are of the form a^{k-1} is called $r-j^{th} overGa_i\ partition$.
- NOTATION: $t_i = \begin{cases} 1+\log_a i & \text{if } \log_a i \text{ is an integer} \\ [1+\log_a i]+1 & \text{if } \log_a i \text{ is not an integer} \end{cases}$

$$r_0 = r, 0 < r_i < r_{i-1}$$
$$n_{i-1} = (r-r_1)\mu_l + (r_1-r_2)\mu_{l-1} ... + (r_{i-3}-r_{i-2})\mu_{l-i+3} + r_{i-2}\mu_{l-i+2}$$
$$\mu_{l-i+1}{}^{(i-1)} = a^{k-1} - \mu_{l-i+1}$$
$$\beta(k) = \beta = \begin{cases} 1 & \text{if } \frac{n}{r} = a^{k-1} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_{i-1} = \begin{cases} 1 & \text{if } \frac{n_{i-1}}{r_{i-1}} = \mu_{l-i+1}{}^{(i-1)} \\ 0 & \text{otherwise} \end{cases}$$
$$\psi_1 = \sum_{k=t_1}^{\infty} \overline{p_{r-1}}^{\,j}\left(n-a^{k-1}r\right),\ \psi_2 = \sum_{k=t_1}^{\infty} \overline{p_{r-1}}^{\,j}\left(n-a^{k-1}-1\right)r-1 \text{ and } \psi_3 = \sum_{k=t_1}^{\infty} \beta(k)$$

$$\psi_{1,i-1} = \sum_{k=t_i}^{\infty}\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\cdots\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{\mu_l=1}^{\mu_{l-1}-1}\sum_{\mu_{l-1}=1}^{\mu_{l-2}-1}\cdots\sum_{\mu_{l-i+2}=1}^{a^{k-1}-1} \overline{p_{r_{i-1}-1}}^{\,j}\left[(n-n_{i-1})-\left(a^{k-1}-\mu_{l-i+2}\right)r_{i-1}\right]$$
$$\psi_{2,i-1} = \sum_{k=t_i}^{\infty}\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\cdots\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{\mu_l=1}^{\mu_{l-1}-1}\sum_{\mu_{l-1}=1}^{\mu_{l-2}-1}\cdots\sum_{\mu_{l-i+2}=1}^{a^{k-1}-1} \overline{p_{r_{i-1}-1}}^{\,j}\left[(n-n_{i-1})-\left(a^{k-1}-\mu_{l-i+2}-1\right)r_{i-1}-1\right]$$
$$\psi_{3,i-1} = \sum_{k=t_i}^{\infty}\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\cdots\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{\mu_l=1}^{\mu_{l-1}-1}\sum_{\mu_{l-1}=1}^{\mu_{l-2}-1}\cdots\sum_{\mu_{l-i+2}=1}^{a^{k-1}-1} \beta_{i-1}$$

SECTION-2 GENERATING FUNCTION

We first prove the following theorem on number of $r - j^{th} overGa_i\ partitions$ n.

2.1 Theorem: If $1 \le r \le n, a \in N$ and $1 \le a^{k-1} \le \left[\frac{n}{r}\right]$, then the number of $r - j^{th} overGa_i\ partitions$ of n with i^{th} smallest part of the form a^{k-1} is

$$i)\ \sum_{k=t_1}^{\infty} \overline{Ga_1\ f_r^{\,1}\left(a^{k-1}, n\right)}^{\,j} = \psi_2 + j.\psi_1 + (j+1)\psi_3$$

$$ii)\ \sum_{k=t_i}^{\infty} \overline{Ga_i\ f_r^{\,i}\left(a^{k-1}, n\right)}^{\,j} = (j+1)^{i-1}\psi_{2,i-1} + j(j+1)^{i-1}\psi_{1,i-1} + (j+1)^{i}\psi_{3,i-1}$$

Proof :

Let $n = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ,..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right)$ be any $r - Ga_1\ partition$ of n with l distinct parts.

Correspondingly there are $(j+1)^l$ times $r - j^{th} overGa\ partitions$ of n.

$$\left[\begin{array}{l} in\ \ theorem\quad put\ \ t=1 \\ \overline{Ga\, spt\left(a^{k-1}, n\right)}^{\,j} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p\left(a^{k-1}, n - ta^{k-1}\right)}^{\,j} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p\left(a^{k-1}+1, n - ta^{k-1}\right)}^{\,j} + (j+1)\sum_{k=1}^{\infty} d(a,n) \end{array}\right]$$

(Put $t=1$ in theorem) We get the number of $r - j^{th} overGa_1\ partitions$ of n with a^{k-1} as first smallest (=smallest) part is

$$\overline{Ga_1\ f_r^{\,1}\left(a^{k-1}, n\right)}^{\,j} = \overline{p\left(a^{k-1}, n - a^{k-1}\right)}^{\,j} + j.\overline{p\left(a^{k-1}+1, n - a^{k-1}\right)}^{\,j} + (j+1)\beta(k)$$

Replace $a^{k-1}+1$ by a^{k-1}, r by $r-1$ and then replace n by $n - a^{k-1}$ in the first sumand of RHS and also replace n by n-a^{k-1} and r by r-1 for in the second.

$$\left[i.e\ \overline{p_r\left(a^{k-1}+1, n\right)}^{\,j} = \overline{p_r\left(n - a^{k-1} r\right)}^{\,j} \right]$$

We get the number of $r - j^{th} overGa_1\ partitions$ of n having first smallest parts a^{k-1} is

$$\overline{Ga_1\ f_r^{\,1}\left(a^{k-1}, n\right)}^{\,j} = \overline{p_{r-1}\left[n - \left(a^{k-1}-1\right)r - 1\right]}^{\,j} + jp_{r-1}.\overline{\left(n - a^{k-1} r\right)}^{\,j} + (j+1)\beta(k) ..... \quad (1)$$

So $\sum_{k=t_1}^{\infty} \overline{Ga_1\ f_r^{\,1}\left(a^{k-1}, n\right)}^{\,j} = \psi_2 + j.\psi_1 + (j+1)\psi_3 \ldots.$ (2)

When $i = 2$, $n = \left(\mu_1^{\alpha_1}, \ldots, \mu_{l-i}^{\alpha_{l-i}}, \mu_{l-i+1}^{\alpha_{l-i+1}}, \mu_{l-i+2}^{\alpha_{l-i+2}}, \ldots, \left(a^{k-1}\right)^{\alpha_{l-1}}, \mu_l^{\alpha_l}\right)$

Subtracting μ_l from each part, we get

$$n_1 = \left(\left(\mu_1^{(1)}\right)^{\alpha_1}, \ldots, \left(\mu_{l-i}^{(1)}\right)^{\alpha_{l-i}}, \left(\mu_{l-i+1}^{(1)}\right)^{\alpha_{l-i+1}}, \left(\mu_{l-i+2}^{(1)}\right)^{\alpha_{l-i+2}}, \ldots, \left(\mu_{l-1}^{(1)}\right)^{\alpha_{l-1}}\right)$$

where $n_1 = n - r\mu_l, r_1 = r - \alpha_l$, $\mu_i^{(1)} = \mu_i - \mu_l \forall i$ and $\mu_{l-1}^{(1)} = a^{k-1} - \mu_l$

From (1) the number of $r_1 - j^{th}$*overpartitions* of n_1 having smallest part $\mu_{l-1}^{\,1}$ is

$$\overline{f_{r_1}^{\,1}\left(\mu_{l-1}^{\,1}, n\right)}^{\,j} == \overline{p_{r_1-1}}^{\,j}\left[n_1 - \left(\mu_{l-1}^{\,1} - 1\right)r_1 - 1\right] + j.\overline{p_{r_1-1}}^{\,j}\left(n_1 - \mu_{l-1}^{\,1} r_1\right) + (j+1)\beta_1 \quad \ldots\ldots \tag{3}$$

The number of $r - j^{th}$*overGa*$_2$ *partitions* of n having second smallest part a^{k-1} is

$$\overline{Ga_2\, f_r^{\,2}\left(a^{k-1}, n\right)}^{\,j} = (j+1)\left\{\overline{p_{r_1-1}}^{\,j}\left[n - r\mu_l - \left(a^{k-1} - \mu_l - 1\right)r_1 - 1\right] + j.\overline{p_{r_1-1}}^{\,j}\left[n - r\mu_l - \left(a^{k-1} - \mu_l\right)r_1\right] + (j+1)\beta_1\right\}$$

Here the part μ_l varies from 1 to $a^{k-1} - 1$ and r_1 varies from 1 to $r - 1$

(if $\mu_l = a^{k-1}$ or $r_1 = r$, the $r - j^{th}$*overGa*$_2$ *partition* does not have l distinct parts.

It contradicts our assumption for $a^{k-1} > \mu_l$.)

From (2) the number of $r - j^{th}$*overGa*$_2$ *partitions* of n whose second smallest parts are a^{k-1} is

$\sum_{k=t_2}^{\infty} \overline{Ga_2\ f_r^{\,2}\left(a^{k-1}, n\right)}^{\,j} = (j+1)\psi_{2,1} + j(j+1)\psi_{1,1} + (j+1)^2\psi_{3,1} \ldots\ldots\ldots$ (4)

When $i = 3$, $n = \left(\mu_1^{\alpha_1}, \ldots, \mu_{l-i}^{\alpha_{l-i}}, \mu_{l-i+1}^{\alpha_{l-i+1}}, \ldots, \left(a^{k-1}\right)^{\alpha_{l-2}}, \mu_{l-1}^{\alpha_{l-1}}, \mu_l^{\alpha_l}\right).$

Subtracting μ_l from each part, we get r_1 partitions of $n_1 = n - r\mu_1$

$$n - r\mu_1 = n_1 = \left(\left(\mu_1^{(1)}\right)^{\alpha_1}, \ldots, \left(\mu_{l-i}^{(1)}\right)^{\alpha_{l-i}}, \left(\mu_{l-i+1}^{(1)}\right)^{\alpha_{l-i+1}}, \ldots, \left(\mu_{l-2}^{(1)}\right)^{\alpha-2}, \left(\mu_{l-1}^{(1)}\right)^{\alpha_{l-1}}\right)$$

where $n_1 = n - r\mu_l, r_1 = r - \alpha_l$, $\mu_i^{(1)} = \mu_i - \mu_l \forall i$ and $\mu_{l-2}^{(1)} = a^{k-1} - \mu_l$

Again subtracting μ_{l-1} from each λ_i for $i=1$ to r_1, we get

From (1) we have the number of $r_2 - j^{th}$ *overpartitions* of n_2 having smallest part $\mu_{l-2}{}^2$ is

$$\overline{f_{r_2}{}^2\left(\mu_{l-2}{}^2, n\right)}^{j} = \overline{p_{r_2-1}}^{j}\left[n_2 - \left(\mu_{l-2}{}^2 - 1\right)r_2 - 1\right] + j.\overline{p_{r_2-1}}^{j}\left(n_2 - \mu_{l-2}{}^2 r_2\right) + (j+1)\beta_2$$

The number of $r - j^{th} overGa_3$ *partitions* of n having third smallest part a^{k-1} is

$$\overline{Ga_3 f_r{}^3\left(a^{k-1}, n\right)}^{j} = (j+1)^2\left\{\overline{p_{r_2-1}}^{j}\left[n - (r-r_1)\mu_l - r_1\mu_{l-1} - \left(a^{k-1} - \mu_{l-1} - 1\right)r_2 - 1\right] + j.\overline{p_{r_2-1}}^{j}\left[n - (r-r_1)\mu_l - r_1\mu_{l-1} - \left(a^{k-1} - \mu_{l-1}\right)r_2\right] + (j+1)\beta_2\right\}$$

From (1) and (2) the number of $r - j^{th} overGa_3$ *partitions* of n with third smallest parts a^{k-1} is

$$\sum_{k=t_3}^{\infty}\overline{Ga_3\ f_r{}^3\left(a^{k-1}, n\right)}^{j} = (j+1)^2\psi_{2,2} + j(j+1)^2\psi_{1,2} + (j+1)^3\psi_{3,2}\ldots\ldots\ldots \tag{5}$$

Repeating this process for i=4 we get:

The number of $r - j^{th} overGa_4$ *partitions* of n having fourth smallest part a^{k-1} is

$$\sum_{k=t_4}^{\infty}\overline{Ga_4\ f_r{}^4\left(a^{k-1}, n\right)}^{j} = (j+1)^3\left\{\overline{p_{r_3-1}}^{j}\left[n - (r-r_1)\mu_l - (r_1-r_2)\mu_{l-1} - r_2\mu_{l-2} - \left(a^{k-1} - \mu_{l-2} - 1\right)r_3 - 1\right] + j.\overline{p_{r_3-1}}^{j}\left[n - (r-r_1)\mu_l - (r_1-r_2)\mu_{l-1} - r_2\mu_{l-2} - \left(a^{k-1} - \mu_{l-2}\right)r_3\right] + (j+1)\beta_3\right\}$$

From (2) the number of $r - Ga_4$ *partitions* of n with fourth smallest parts a^{k-1} is

$$\sum_{k=t_4}^{\infty}\overline{Ga_4\ f_r{}^4\left(a^{k-1}, n\right)}^{j} = (j+1)^3\psi_{2,3} + j(j+1)^3\psi_{1,3} + (j+1)^4\psi_{3,3}\ldots.. \tag{6}$$

The same process can be repeated in the general case when

$n=\left(\mu_1^{\alpha_1},...,\mu_{l-i}^{\alpha_{l-i}},\left(a^{k-1}\right)^{\alpha_{l-i+1}},\mu_{l-i+2}^{\alpha_{l-i+2}},...,\mu_{l-1}^{\alpha_{l-1}},\mu_l^{\alpha_l}\right)$ is a $r-j^{th}overGa_i\ partition$ of n with l distinct parts and the i^{th} smallest parts of the form a^{k-1}.We get:

The number of $r-j^{th}overGa_i\ partitions$ of n having i^{th} smallest part a^{k-1} is

$$\sum_{k=t_i}^{\infty}\overline{Ga_i\ f_r^{\ i}\left(a^{k-1},n\right)}^{\,j}=\left(j+1\right)^{i-1}\psi_{2,i-1}+j\left(j+1\right)^{i-1}\psi_{1,i-1}+\left(j+1\right)^{i}\psi_{3,i-1}.........\qquad(7)$$

Hence the theorem

The next theorem yields a generating function for the number of i^{th} smallest parts of $r-j^{th}overGa_i\ partitions$ of n where i^{th} smallest part is the first part.

2.2 Theorem: The generating function for the number of i^{th} smallest parts of $r-j^{th}overGa_i\ partitions$ of n in which parts are distinct and i^{th} smallest part is the first part is given by

$$\left(j+1\right)^{i}\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\sum_{r_3=1}^{r_2-1}\sum_{r_4=1}^{r_3-1}...\sum_{r_{i-1}=1}^{r_{i-2}-1}\left(\sum_{\mu_i=1}^{\mu_{i-1}-1}q^{(r-r_1)\mu_i}\sum_{\mu_{i-1}=1}^{\mu_{i-2}-1}q^{(r_1-r_2)\mu_{i-1}}...\sum_{\mu_2=1}^{a^{k-1}-1}q^{(r_{i-2}-r_{i-1})\mu_2}\sum_{k=t_i}^{\infty}q^{a^{k-1}r_{i-1}}\right)$$

Proof: Let $n=\left(a^{k-1}\right)$.

We know from (2) and (1) that the number of smallest parts of $r-Ga_1\ partitions$ of n such that smallest part is the first part which is a^{k-1} is 1 if $\frac{n}{r}=a^{k-1}$, 0 otherwise and has the generating function $\sum_{k=t_1}^{\infty}q^{a^{k-1}}$.

Therefore, the number of smallest parts of $r-j^{th}overGa_1\ partitions$ of n such that smallest part is the first part a^{k-1} is given by

$$\left(j+1\right)\sum_{k=t_1}^{\infty}q^{a^{k-1}}\quad ....\qquad(8)$$

Let $n=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2}\right)$ be any $r-Ga_2\ partition$ of n with two distinct parts and second smallest part is a^{k-1}. Each $r-Ga_2\ partition$ has $\left(j+1\right)^2\left(r-j^{th}overGa\ partitions\right)$.

Subtracting μ_2 from each part, we get

$n_1=\left(\mu_1^{1}\right)^{\alpha_1}$ where $n_1=n-r\mu_2$, $r_1=r-\alpha_2$ and $\mu_1^{1}=\mu_1-\mu_2=a^{k-1}-\mu_2$.

The number of smallest parts of $r_1 - j^{th} overpartitions$ of n_1 such that smallest part is the first part and having $\mu_1^{\,1}$ as smallest part is $1 \quad \text{if} \quad \frac{n_1}{r_1} = a^{k-1}, \;\; 0 \quad \text{otherwise}$.

Hence the number of second smallest parts of $r - Ga_2\ partitions$ of n such that second smallest part is the first part and having a^{k-1} is a smallest part is $1 \quad \text{if} \quad \frac{n_1}{r_1} = a^{k-1}$,

$0 \quad \text{otherwise}$ and from (2) and (1) its generating function is

$$\sum_{\mu_2=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{k=t_2}^{\infty} q^{r\mu_2+\left(a^{k-1}-\mu_2\right)r_1}$$

$$= \sum_{\mu_2=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{k=t_2}^{\infty} q^{(r-r_1)\mu_2+a^{k-1}r_1}$$

$$= \sum_{r_1=1}^{r-1}\left[\sum_{\mu_2=1}^{a^{k-1}-1} q^{r\mu_2}\sum_{k=t_2}^{\infty} q^{\left(a^{k-1}-\mu_2\right)r_1}\right]$$

Consequently, the number of second smallest parts of $r - j^{th} overGa_2\ partitions$ of n such that second smallest part is the first part a^{k-1} is

$$\left(j+1\right)^2 \sum_{r_1=1}^{r-1}\left[\sum_{\mu_2=1}^{a^{k-1}-1} q^{r\mu_2}\sum_{k=t_2}^{\infty} q^{\left(a^{k-1}-\mu_2\right)r_1}\right] \;\; \ldots\ldots.. \qquad (9)$$

Let $n = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \mu_3^{\alpha_3}\right)$ be any $r - Ga_3\ partition$ of n with three distinct parts and third smallest part is a^{k-1}. Correspondingly there are $\left(j+1\right)^3$ times $r - j^{th} overGa\ partitions$ of n.

Subtracting μ_3 from each λ_i for $i = 1$ to r, we get

$n_1 = \left(\left(\mu_1^{(1)}\right)^{\alpha_1}, \left(\mu_2^{(1)}\right)^{\alpha_2}\right)$ where $n_1 = n - r\mu_3,\;\; r_1 = r - \alpha_3$ and $\mu_i^{(1)} = \mu_i - \mu_3$

Again subtracting $\left(\mu_2^{(1)}\right)$ from each λ_i for $i = 1$ to r_1, we get

$$n_2 = \left(\mu_1^{(2)}\right)^{\alpha_1} \quad \text{where} \quad n_2 = n_1 - r_1\mu_2^{(1)} = n - r\mu_3 - r_1\left(\mu_2 - \mu_3\right) = n - r\mu_3 - r_1\mu_2 + r_1\mu_3$$

$$= n - \left(r - r_1\right)\mu_3 - r_1\mu_2$$

$r_2 = r_1 - \alpha_2 = r - \alpha_1 - \alpha_2$ and

$\mu_1^{(2)} = \mu_1^{(1)} - \mu_2^{(1)} = \left(\mu_1 - \mu_3\right) - \left(\mu_2 - \mu_3\right) = \mu_1 - \mu_2 = a^{k-1} - \mu_2$

The number of smallest parts of $r_2 - partitions$ of n_2 such that smallest part is the first part and having $\mu_1^{(2)}$ is a smallest part is 1 if $\frac{n_2}{r_2} = a^{k-1}$, 0 otherwise.

Hence the number of third smallest parts of $r - Ga_3\ partitions$ of n such that third smallest part is the first part and having a^{k-1} is a smallest part is 1 if $\frac{n_2}{r_2} = a^{k-1}$, 0 otherwise and from (2) and (1) the generating function is given by

$$\sum_{k=t_3}^{\infty}\sum_{r_2=1}^{r_1-1}\sum_{\mu_2=1}^{a^{k-1}-1}\sum_{\mu_3=1}^{\infty}\sum_{r_1=1}^{r-1} q^{(r-r_1)\mu_3+(r_1-r_2)\mu_2+a^{k-1}r_2}$$

$$= \sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\left(\sum_{\mu_3=1}^{\mu_2-1} q^{(r-r_1)\mu_3}\sum_{\mu_2=1}^{a^{k-1}-1} q^{(r_1-r_2)\mu_2}\sum_{k=t_3}^{\infty} q^{a^{k-1}r_2}\right)$$

The generating function for number of third smallest parts of $r - j^{th}overGa_3\ partitions$ of n such that third smallest part is the first part a^{k-1} is

$$(j+1)^3\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\left(\sum_{\mu_3=1}^{\mu_2-1} q^{(r-r_1)\mu_3}\sum_{\mu_2=1}^{a^{k-1}-1} q^{(r_1-r_2)\mu_2}\sum_{k=t_3}^{\infty} q^{a^{k-1}r_2}\right) \text{........} \qquad (10)$$

The same process extends to any i.

Let $n = (\lambda_1, \lambda_2, ..., \lambda_r) = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{i-1}^{\alpha_{i-1}}, \mu_i^{\alpha_i}\right)$ be any $r - Ga_i\ partition$ of n with i distinct parts and i^{th} smallest part is a^{k-1}. Correspondingly there are $(j+1)^i$ times $r - j^{th}overGa\ partitions$ of n.

Subtracting μ_i from each λ_i for $i = 1$ to r, we get

$$n_1 = \left(\left(\mu_1^{(1)}\right)^{\alpha_1}, \left(\mu_2^{(1)}\right)^{\alpha_2}, ..., \left(\mu_{i-1}^{(1)}\right)^{\alpha_{i-1}}\right) \text{ where } n_1 = n - r\mu_i, \quad r_1 = r - \alpha_i$$

$$\text{and} \quad \mu_{i_1}^{(1)} = \mu_{i_1} - \mu_i$$

Again subtracting $\left(\mu_{i-1}^{(1)}\right)$ from each λ_i for $i = 1$ to r_1, we get

$$n_2 = \left(\left(\mu_1^{(2)}\right)^{\alpha_1}, \left(\mu_2^{(2)}\right)^{\alpha_2}, ..., \left(\mu_{i-2}^{(2)}\right)^{\alpha_{i-2}}\right) \text{ where } n_2 = n_1 - r_1\mu_{i-1}^{(1)}, r_2 = r_1 - \alpha_{i-1}$$

$$\text{and } \mu_{i_2}^{(2)} = \mu_{i_2}^{(1)} - \mu_{i-1}^{(1)}$$

Continue this process finally we get

$n_{i-1} = \left(\left(\mu_1^{(i-1)}\right)^{\alpha_1}\right)$ where $n_{i-1} = n_{i-2} - r_{i-2}\mu_{i-2}^{(i-2)}, r_{i-1} = r_{i-2} - \alpha_{i-1}$

and $\mu_{i_{i-1}}^{(i-1)} = \mu_{i_{i-1}}^{(i-2)} - \mu_{i-2}^{(i-2)}$

The number of smallest parts of $r_{i-1} - partitions$ of n_{i-1} such that smallest part is the first part and having $\mu_1^{(i-1)}$ as the smallest part is 1 if $\frac{n_{i-1}}{r_{i-1}} = a^{k-1}$, 0 otherwise.

Hence the number of i^{th} smallest parts of $r - Ga_i\ partitions$ of n such that i^{th} smallest part is the first part and having a^{k-1} as the smallest part is 1 if $\frac{n_{i-1}}{r_{i-1}} = a^{k-1}$, 0 otherwise and from (2)and (1) the generating function for number of i^{th} smallest parts of $r - Ga_i\ partitions$ of n such that i^{th} smallest part as the first part a^{k-1} is

$$\sum_{k=t_i}^{\infty}\sum_{r_2=1}^{r_1-1}\sum_{\mu_i=1}^{\mu_{i-1}-1}\sum_{\mu_{i-1}=1}^{\mu_{i-2}-1}\sum_{r_1=1}^{r-1}\sum_{\mu_2=1}^{a^{k-1}-1}\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{r_{i-1}=1}^{r_{i-2}-1} q^{r\mu_i + r_1(\mu_{i-1}-\mu_i)+\ldots+r_{i-2}(\mu_{i-2}-\mu_{i-1})+\left(a^{k-1}-\mu_{i-1}\right)r_{i-1}}$$

$$= \sum_{k=t_i}^{\infty}\sum_{r_2=1}^{r_1-1}\sum_{\mu_i=1}^{\mu_{i-1}-1}\sum_{\mu_{i-1}=1}^{\mu_{i-2}-1}\sum_{r_1=1}^{r-1}\sum_{\mu_2=1}^{a^{k-1}-1}\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{r_{i-1}=1}^{r_{i-2}-1} q^{(r-r_1)\mu_i+(r_1-r_2)\mu_{i-1}+\ldots+(r_{i-2}-r_{i-1})\mu_2+a^{k-1}r_{i-1}}$$

$$= \sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\sum_{r_3=1}^{r_2-1}\sum_{r_4=1}^{r_3-1}\cdots\sum_{r_{i-1}=1}^{r_{i-2}-1}\left(\sum_{\mu_i=1}^{\mu_{i-1}-1} q^{(r-r_1)\mu_i}\sum_{\mu_{i-1}=1}^{\mu_{i-2}-1} q^{(r_1-r_2)\mu_{i-1}}\cdots\sum_{\mu_2=1}^{a^{k-1}-1} q^{(r_{i-2}-r_{i-1})\mu_2}\sum_{k=t_i}^{\infty} q^{a^{k-1}r_{i-1}}\right)$$

The generating function for number of i^{th} smallest parts of $r - j^{th} overGa_i\ partitions$ of n such that i^{th} smallest part as the first part a^{k-1} is

$$(j+1)^i \sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\sum_{r_3=1}^{r_2-1}\sum_{r_4=1}^{r_3-1}\cdots\sum_{r_{i-1}=1}^{r_{i-2}-1}\left(\sum_{\mu_i=1}^{\mu_{i-1}-1} q^{(r-r_1)\mu_i}\sum_{\mu_{i-1}=1}^{\mu_{i-2}-1} q^{(r_1-r_2)\mu_{i-1}}\cdots\sum_{\mu_2=1}^{a^{k-1}-1} q^{(r_{i-2}-r_{i-1})\mu_2}\sum_{k=t_i}^{\infty} q^{a^{k-1}r_{i-1}}\right) \quad (11)$$

2.3 Illustration: Taking $r = 3, a = 2, j = 2\ and\ i = 3$ the generating function is

$$(2+1)^3 \sum_{\mu_3=1}^{\mu_2-1}\sum_{r_1=1}^{r-1} q^{(r-r_1)\mu_3}\sum_{r_1=1}^{r-1}\sum_{r_2=1}^{r_1-1}\sum_{\mu_2=1}^{a^{k-1}-1} q^{(r_1-r_2)\mu_2}\sum_{r_2=1}^{r_1-1}\sum_{k=t_3}^{\infty} q^{2^{k-1}r_2}$$

$$= 27\sum_{\mu_3=1}^{\mu_2-1}\sum_{r_1=1}^{2} q^{(3-r_1)\mu_3}\sum_{r_1=1}^{2}\sum_{r_2=1}^{r_1-1}\sum_{\mu_2=1}^{2^{k-1}-1} q^{(r_1-r_2)\mu_2}\sum_{r_2=1}^{r_1-1}\sum_{k=3}^{\infty} q^{2^{k-1}r_2}$$

$$= 27\sum_{\mu_3=1}^{\mu_2-1} q^{\mu_3} \sum_{r_2=1}^{1} \sum_{\mu_2=1}^{2^{k-1}-1} q^{(2-r_2)\mu_2} \sum_{r_2=1}^{1}\sum_{k=3}^{\infty} q^{2^{k-1}r_2}$$

$$= 27\sum_{\mu_3=1}^{\mu_2-1} q^{\mu_3} \sum_{\mu_2=1}^{2^{k-1}-1} q^{\mu_2} \sum_{k=3}^{\infty} q^{2^{k-1}}$$

$$= 27\sum_{k=3}^{\infty} q^{2^{k-1}} \sum_{\mu_2=1}^{2^{k-1}-1} q^{\mu_2} \sum_{\mu_3=1}^{\mu_2-1} q^{\mu_3}$$

$$= 27q^4 \sum_{\mu_2=1}^{3} q^{\mu_2} \sum_{\mu_3=1}^{\mu_2-1} q^{\mu_3} + q^8 \sum_{\mu_2=1}^{7} q^{\mu_2} \sum_{\mu_3=1}^{\mu_2-1} q^{\mu_3} + \ldots$$

$$= 27\left(q^7 + q^8 + q^9 + q^{11} + q^{12} + 2q^{13} + 2q^{14} + 3q^{15} + 3q^{16} + 4q^{17} + 4q^{18} + \ldots\right)$$

$$= 27q^7 + 27q^8 + 27q^9 + 27q^{11} + 27q^{12} + 54q^{13} + 54q^{14} + 81q^{15} + 81q^{16} + 108q^{17} + 108q^{18} + \ldots$$

When $n = 15$ the number of third smallest parts of second over $G2_3$ partitions of 15 are used by the three partitions $8 + 6 + 1$, $8 + 5 + 2$ and $8 + 4 + 3$ each of these three partitions induces 27 over partitions and are listed below. Also 81 is coefficient of q^{15}.

8	6	1	8	6	$\bar{1}$	8	6	$\bar{\bar{1}}$
$\bar{8}$	6	1	$\bar{8}$	6	$\bar{1}$	$\bar{8}$	6	$\bar{\bar{1}}$
$\bar{\bar{8}}$	6	1	$\bar{\bar{8}}$	6	$\bar{1}$	$\bar{\bar{8}}$	6	$\bar{\bar{1}}$
8	$\bar{6}$	1	8	$\bar{6}$	$\bar{1}$	8	$\bar{6}$	$\bar{\bar{1}}$
$\bar{8}$	$\bar{6}$	1	$\bar{8}$	$\bar{6}$	$\bar{1}$	$\bar{8}$	$\bar{6}$	$\bar{\bar{1}}$
$\bar{\bar{8}}$	$\bar{6}$	1	$\bar{\bar{8}}$	$\bar{6}$	$\bar{1}$	$\bar{\bar{8}}$	$\bar{6}$	$\bar{\bar{1}}$
8	$\bar{\bar{6}}$	1	8	$\bar{\bar{6}}$	$\bar{1}$	8	$\bar{\bar{6}}$	$\bar{\bar{1}}$
$\bar{8}$	$\bar{\bar{6}}$	1	$\bar{8}$	$\bar{\bar{6}}$	$\bar{1}$	$\bar{8}$	$\bar{\bar{6}}$	$\bar{\bar{1}}$
$\bar{\bar{8}}$	$\bar{\bar{6}}$	1	$\bar{\bar{8}}$	$\bar{\bar{6}}$	$\bar{1}$	$\bar{\bar{8}}$	$\bar{\bar{6}}$	$\bar{\bar{1}}$

8	5	2	8	5	$\bar{2}$	8	5	$\bar{\bar{2}}$
$\bar{8}$	5	2	$\bar{8}$	5	$\bar{2}$	$\bar{8}$	5	$\bar{\bar{2}}$
$\bar{\bar{8}}$	5	2	$\bar{\bar{8}}$	5	$\bar{2}$	$\bar{\bar{8}}$	5	$\bar{\bar{2}}$
8	$\bar{5}$	2	8	$\bar{5}$	$\bar{2}$	8	$\bar{5}$	$\bar{\bar{2}}$
$\bar{8}$	$\bar{5}$	2	$\bar{8}$	$\bar{5}$	$\bar{2}$	$\bar{8}$	$\bar{5}$	$\bar{\bar{2}}$
$\bar{\bar{8}}$	$\bar{5}$	2	$\bar{\bar{8}}$	$\bar{5}$	$\bar{2}$	$\bar{\bar{8}}$	$\bar{5}$	$\bar{\bar{2}}$
8	$\bar{\bar{5}}$	2	8	$\bar{\bar{5}}$	$\bar{2}$	8	$\bar{\bar{5}}$	$\bar{\bar{2}}$
$\bar{8}$	$\bar{\bar{5}}$	2	$\bar{8}$	$\bar{\bar{5}}$	$\bar{2}$	$\bar{8}$	$\bar{\bar{5}}$	$\bar{\bar{2}}$
$\bar{\bar{8}}$	$\bar{\bar{5}}$	2	$\bar{\bar{8}}$	$\bar{\bar{5}}$	$\bar{2}$	$\bar{\bar{8}}$	$\bar{\bar{6}}$	$\bar{\bar{2}}$

8	4	3	8	4	$\bar{3}$	8	4	$\bar{\bar{3}}$
$\bar{8}$	4	3	$\bar{8}$	4	$\bar{3}$	$\bar{8}$	4	$\bar{\bar{3}}$
$\bar{\bar{8}}$	4	3	$\bar{\bar{8}}$	4	$\bar{3}$	$\bar{\bar{8}}$	4	$\bar{\bar{3}}$
8	$\bar{4}$	3	8	$\bar{4}$	$\bar{3}$	8	$\bar{4}$	$\bar{\bar{3}}$
$\bar{8}$	$\bar{4}$	3	$\bar{8}$	$\bar{4}$	$\bar{3}$	$\bar{8}$	$\bar{4}$	$\bar{\bar{3}}$
$\bar{\bar{8}}$	$\bar{4}$	3	$\bar{\bar{8}}$	$\bar{4}$	$\bar{3}$	$\bar{\bar{8}}$	$\bar{4}$	$\bar{\bar{3}}$
8	$\bar{\bar{4}}$	3	8	$\bar{\bar{4}}$	$\bar{3}$	8	$\bar{\bar{4}}$	$\bar{\bar{3}}$
$\bar{8}$	$\bar{\bar{4}}$	3	$\bar{8}$	$\bar{\bar{4}}$	$\bar{3}$	$\bar{8}$	$\bar{\bar{4}}$	$\bar{\bar{3}}$
$\bar{\bar{8}}$	$\bar{\bar{4}}$	3	$\bar{\bar{8}}$	$\bar{\bar{4}}$	$\bar{3}$	$\bar{\bar{8}}$	$\bar{\bar{4}}$	$\bar{\bar{3}}$

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