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On Fixed Point Theorems in Complete Metric Space

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ABSTRACT

In this paper, some theorems concerning the existence and uniqueness of fixed point in complete metric space are established. The results of the continuity clause we reached are introduced and some of the results we obtained are circulated.

Keywords: Complete metric space, Banach fixed point, Continuous mapping.

1. INTRODUCTION

Fixed point theory remains a very powerful and polar tool in modern mathematics especially in existence and uniqueness considerations. The first results related to this topic appeared in the year 1922 by the Polish mathematician S. Banash[1]. The fixed point theory of mapping has been developed in metric space by many authors (see[4,5,13]). Further, many mathematics worked on a contraction mappings and Banach contraction principle (see[3,9,12]). R. Kannan in[11] proved that if T is a self-mapping of complete metric space X into itself satisfying $d(Tx,Ty) \leq \alpha[d(Tx,x) + d(Ty,y)]$ for all $x,y \in X$ where $\alpha \in (0,\frac{1}{2})$. (1.1)

Then T has unique fixed point in X and B. Fisher[7] proved the result with $d(Tx,Ty) \leq \alpha[d(Tx,y) + d(Ty,x)]$ for all $x,y \in X$ where $\alpha \in (0,\frac{1}{2})$. (1.2)

A similar conclusion was also obtained by R. Bhardwaj *et al*.[2], S. K. Chaterjee[6], G. E. Hardy *et al*.[10] and D. P. Shukla *et al*.[14].

In this paper we will prove some theorems on the existence and unique of the fixed point in complete metric space.

2. MAIN RESULTS

Theorem 2.1 Let X be a complete metric space and let $f: X \to X$ a continuous self-mapping on X, suppose f satisfying the condition

$$d\big(f(x), f(y)\big) \leq m_1 d\big(x, f(x)\big) + m_2 d\big(y, f(y)\big) + m_3 d\big(x, f(y)\big) + m_4 d(x, y) \qquad (2.1)$$

for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ such that $\sum_{i=1}^{4} m_i < 1$. Then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X and $\{x_{n-1}\}_{n=1}^{\infty}$ be the sequence of iterations for f at x_0, such that $f(x_{n-1}) = x_n$ (2.2)

We let that $x_{n-1} \neq x_n$ for all $n \in N$. Therefore $d(x_{n-1}, x_n) = d(f(x_{n-2}), f(x_{n-1}))$.

So, $d(x_{n-1}, x_n) \leq m_1 d\big(x_{n-2}, f(x_{n-2})\big) + m_2 d(x_{n-1}, f(x_{n-1})) + m_3 d\big(x_{n-2}, f(x_{n-1})\big) + {}$ $+ m_4 d(x_{n-2}, x_{n-1})$. And by (2.2) we find that,

$d(x_{n-1}, x_n) \leq m_1 d(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_n) + m_4 d(x_{n-2}, x_{n-1})$.

By triangle inequality for some $x_{n-2} \leq x_{n-1} \leq x_n$, we obtained

$$d(x_{n-1}, x_n) \leq m_1 d(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_{n-1}) + m_3 d(x_{n-1}, x_n) + m_4 d(x_{n-2}, x_{n-1}).$$

$$= \left(\frac{m_1 + m_3 + m_4}{1 - m_2 - m_4}\right) d(x_{n-2}, x_{n-1}) \qquad (2.3)$$

And, $d(x_{n-1}, x_n) \leq \left(\frac{m_1 + m_3 + m_4}{1 - m_2 - m_3}\right)^2 d(x_{n-3}, x_{n-2})$. So, if we repeat this work we obtain

$$d(x_{n-1}, x_n) \leq \left(\frac{m_1 + m_3 + m_4}{1 - m_2 - m_3}\right)^n d(x_0, x_1) \qquad (2.4)$$

For some $s \geq n - 1$, we have

$d(x_{n-1}, x_s) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \cdots + d(x_{s-1}, x_s)$ by (2.4) we conclude that $d(x_{n-1}, x_s) \leq \{\beta^n + \beta^{n+1} + \cdots + \beta^s\} d(x_0, x_1)$, where $\beta = \left(\frac{m_1 + m_3 + m_4}{1 - m_2 - m_3}\right)$, and since $\sum_{i=1}^{4} m_i < 1$. Therefore $\beta^n \to 0 \ as\ n \to \infty$. Then, $d(x_{n-1}, x_s) \to 0 \ as\ s \to \infty$.

Every Cauchy sequence $\{x_{n-1}\}_{n=1}^{\infty}$ in X is convergence, since X is a complete space. i.e. there exist $z_1 \in X$ such that $x_n \to z_1$, also we have a continuous self-mapping, then

$$f\left(\lim_{n\to\infty} x_n\right) = f(z_1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z_1$$

Hence, z_1 is a fixed point of f in X.

We need to prove that z_1 is a unique fixed point of f in X, for that we let that there exists another fixed point $z_2 \in X$, such that $z_1 \neq z_2$ and $f(z_1) = z_1$, $f(z_2) = z_2$, therefore by (2.1),

$d(z_1, z_2) = d\big(f(z_1), f(z_2)\big) \leq m_1 d\big(z_1, f(z_1)\big) + m_2 d\big(z_2, f(z_2)\big) + m_3 d\big(z_1, f(z_2)\big) + {}$ $+ m_4 d(z_1, z_2)$.

$d(z_1, z_2) \leq (m_3 + m_4) d(z_1, z_2)$, $(m_3 + m_4) < 1$.

Then $d(z_1, z_2) \leq 0$, which implies that $d(z_1, z_2) = 0$, so $z_1 = z_2$ and z_1 is a unique fixed point of f in X.

Theorem 2.2 Let X be a complete metric space and let f_1, f_2 two continuous self-mappings on X, satisfying the condition

$$d\big(f_1(x), f_2(y)\big) \le m_1 d\big(x, f_1(x)\big) + m_2 d\big(y, f_2(y)\big) + m_3 d\big(x, f_2(y)\big) + m_4 d(x, y) \tag{2.5}$$

for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ such that $\sum_{i=1}^{4} m_i < 1$. Then f_1 and f_2 have a unique common fixed point.

Proof. For $x_0 \in X$, $y_0 \in X$ take $f_1(x_{k-1}) = x_n$, $f_2(y_{k-1}) = y_n$. So,

$$d(x_k, y_k) = d\big(f_1(x_{k-1}), f_2(y_{k-1})\big) \le m_1 d\big(x_{k-1}, f_1(x_{k-1})\big) + m_2 d\big(y_{k-1}, f_2(y_{k-1})\big) + + m_3 d\big(x_{k-1}, f_2(y_{k-1})\big) + m_4 d(x_{k-1}, y_{k-1}).$$

$$= m_1 d(x_{k-1}, x_k) + m_2 d(y_{k-1}, y_k) + m_3 d(x_{k-1}, y_k) + m_4 d(x_{k-1}, y_{k-1}).$$

Also we have

$$\sum_{k=1}^{n} d(x_k, y_k) = \sum_{k=1}^{n} d\big(f_1(x_{k-1}), f_2(y_{k-1})\big) \le \sum_{k=1}^{n} [m_1 d(x_{k-1}, x_k) + m_2 d(y_{k-1}, y_k) + m_3 d(x_{k-1}, y_k) + m_4 d(x_{k-1}, y_{k-1})].$$

$$\sum_{k=1}^{n} d(x_k, y_k) \le\le [m_1 d(x_0, x_n) + m_2 d(y_0, y_n) + m_3 \sum_{k=1}^{n} d(x_{k-1}, y_k) + m_4 \sum_{k=1}^{n} d(x_{k-1}, y_{k-1})].$$

And, $\sum_{k=1}^{n} d(x_{k+1}, y_k) \le [m_1 d(x_1, x_n) + m_2 d(y_0, y_n) + m_3 \sum_{k=1}^{n} d(x_k, y_k) + + m_4 \sum_{k=1}^{n} d(x_k, y_{k-1})].$

Also, $\sum_{k=1}^{n} d(x_k, x_{k+1}) \le (m_1 + m_4) d(x_0, x_n) + (m_2 + m_3) d(x_1, x_{n+1})$.

We get that $\sum_{k=1}^{n} d(x_k, x_{k+1}) \le \sum_{k=1}^{n} d(x_k, y_k) + \sum_{k=1}^{n} d(y_k, x_{k+1})$.

Hence, $\sum_{k=1}^{n} d(x_k, x_{k+1}) < \infty$. (2.6)

This implies that $d(x_k, x_{k+1}) \to 0$ as $k \to \infty$, therefore we see that $\{x_k\}$ is a Cauchy sequence in X. Similarly, we can show that $\{y_k\}$ is a Cauchy sequence in X, and since X is a complete metric space, so there exists a common fixed point in X,

Let $z_1 = \lim_{n\to\infty} x_n$, $z_2 = \lim_{n\to\infty} y_n$, for all $z_1, z_2 \in X$.

This implies that

$$d(x_n, z_1) \to 0 \;, \quad n \to \infty$$
$$d(y_n, z_2) \to 0 \;, \quad n \to \infty$$

As f_1, f_2 are continuous mapping, we get

$$d(f_1(x_n), f_1(z_1)) \to 0 \;, \quad n \to \infty$$
$$d(f_2(y_n), f_2(z_2)) \to 0 \;, \quad n \to \infty$$

This means that

$$d\big(z_1, f_1(z_1)\big) = d(f_1^{-1}(f_1(z_1), f_1(z_1)) \le m_1 d(f_1(z_1), f_1^{-1}\big(f_1(z_1)\big) + m_2 d(z_1, f_1(z_1)) + + m_3 d(f_1(z_1), f_1(z_1)) + m_4 d(f_1(z_1), z_1) \tag{2.7}$$

$$= m_1 d(f_1(z_1), z_1) + m_2 d\big(z_1, f_1(z_1)\big) + m_4 d(f_1(z_1), z_1).$$

$$= (m_1 + m_2 + m_4)\, d\big(z_1, f_1(z_1)\big).$$

Hence, $f_1(z_1) = z_1$. Similarly, we can show that $f_2(z_2) = z_2$.

Now, for each $m_1, m_2, m_3, m_4 \in [0,1)$ and $z_1, z_2 \in X$, we get

$$d(z_1, z_2) = d\big(f_1(z_1), f_2(z_2)\big) \le m_1 d\big(z_1, f_1(z_1)\big) + m_2 d\big(z_2, f_2(z_2)\big) + m_3 d\big(z_1, f_2(z_2)\big) + + m_4 d(z_1, z_2).$$

$$d(z_1, z_2) \le m_1 d(z_1, z_1) + m_2 d(z_2, z_2) + m_3 d(z_1, z_2) + m_4 d(z_1, z_2).$$

$$= (m_3 + m_4) d(z_1, z_2)$$

Therefore, $d(z_1, z_2) \leq 0$.
Hence, $z_1 = z_2$. This implies that z_1 is common fixed point of f_1 and f_2 in X.
Now, let $z_3 \in X$ be another fixed point of f_1 and f_2 in X such that
$f_1(z_3) = z_3$ and $f_2(z_3) = z_3$.
Therefore,

$$d(z_1, z_3) = d\big(f_1(z_1), f_2(z_3)\big) \leq m_1 d\big(z_1, f_1(z_1)\big) + m_2 d\big(z_3, f_2(z_3)\big) + m_3 d\big(z_1, f_2(z_3)\big) + \\ + m_4 d(z_1, z_3).$$

$$d(z_1, z_3) \leq m_1 d(z_1, z_1) + m_2 d(z_3, z_3) + m_3 d(z_1, z_3) + m_4 d(z_1, z_3).$$
$$= (m_3 + m_4) d(z_1, z_3).$$

We get that $d(z_1, z_3) \leq 0$. This implies that $z_1 = z_2 = z_3$. Thus z_1 is the unique fixed point of f_1 and f_2 in X.
In the next theorem, we will generalize Theorem 2.1 and 2.2 .

Theorem 2.3 Let $\{f_\propto\}_{\propto \in \Delta}$ be a family continuous self-mapping in complete metric space X, suppose that

$$d\left(f_\propto(x), f_\beta(y)\right) \leq m_1 d\big(x, f_\propto(x)\big) + m_2 d\left(y, f_\beta(y)\right) + m_3 d\left(x, f_\beta(y)\right) + m_4 d(x, y) \quad (2.8)$$

for every $x, y \in X$, $x \neq y$ and $m_1, m_2, m_3, m_4 \in [0,1)$, $\sum_{i=1}^{4} m_i < 1$.
Then there exist a unique $z_1 \in X$ satisfies $f_\propto(z_1) = z_1$ for all $\propto \in \Delta$.

Proof. If we repeat the same work in Theorem 2.2 but we must replace f_1 and f_2 by $f_\propto$ and f_β respectively, we will get a unique point $z_1 \in X$ which satisfies $f_\propto(z_1) = f_\beta(z_1) = z_1$.

In the next theorem we will study the existences and uniqueness of a common fixed point of two mappings which are not necessarily continuous.

Theorem 2.4 Let f_1 and f_2 be two self-mappings on a complete metric space X satisfies

$$d\big(f_1(x), f_2(y)\big) \leq m_1 d\big(x, f_1(x)\big) + m_2 d\big(y, f_2(y)\big) + m_3 d\big(x, f_2(y)\big) + m_4 d(x, y)$$

for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ such that $\sum_{i=1}^{4} m_i < 1$. Suppose that $f_1 f_2 = f_2 f_1$ is continuous then f_1 and f_2 having unique common fixed point in X .

Proof. Take $x_n = f_1(x_{n-1})$, $x_n = f_2(x_{n-1})$ and $f_1(x_{n-1}) \neq f_2(x_{n-1})$, $x_n \neq x_{n-1}$, $\forall n \in N$. Therefore,

$$d(x_{2n+1}, x_{2n}) = d\big(f_1(x_{2n}), f_2(x_{2n-1})\big) \leq m_1 d\big(x_{2n}, f_1(x_{2n})\big) + m_2 d\big(x_{2n-1}, f_2(x_{2n-1})\big) + \\ + m_3 d\big(x_{2n}, f_2(x_{2n-1})\big) + m_4 d(x_{2n}, x_{2n-1}). \quad (2.9)$$
$$= m_1 d(x_{2n}, x_{2n+1}) + m_2 d(x_{2n-1}, x_{2n}) + m_3 d(x_{2n}, x_{2n}) + m_4 d(x_{2n}, x_{2n-1}).$$

$$d(x_{2n+1}, x_{2n}) \leq \left(\frac{m_2 + m_4}{1 - m_2}\right) d(x_{2n}, x_{2n-1}) \quad (2.10)$$

by repeating this work, we get

$$d(x_{2n+1}, x_{2n}) \leq \left(\frac{m_2 + m_4}{1 - m_2}\right)^{2n} d(x_1, x_0) \quad (2.11)$$

Similarly, $d(x_{2n+2}, x_{2n+1}) \leq \left(\frac{m_2 + m_4}{1 - m_2}\right)^{2n+1} d(x_1, x_0) \quad (2.12)$

since x_n is a Cauchy sequence let, $x_n \to z_1$, $n \to \infty$.

Then, $x_{n_k} \to z_1$, $k \to \infty$. So, we have, $f_1 f_2(z_1) = f_2 f_1(z_1) = f_1 f_2(\lim_{k\to\infty} x_{n_k})$

$$= \lim_{k\to\infty} x_{n_{k+1}} = z_1.$$

Let z_1 is a fixed point of $f_1 f_2$ in X i.e. $f_1 f_2(z_1) = z_1$, so we must show that

$$f_1(z_1) = z_1 \text{ and } f_2(z_1) = z_1.$$

Further suppose that $f_1(z_1) \neq z_1$ and $f_2(z_1) \neq z_1$,

So, $d\big(z_1, f_1(z_1)\big) = d\big(f_2 f_1(z_1), f_1(z_1)\big) \leq m_1 d\big(f_1(z_1), f_2 f_1(z_1)\big) + m_2 d\big(z_1, f_1(z_1)\big) +$
$+ m_3 d\big(f_1(z_1), f_1(z_1)\big) + m_4 d(f_1(z_1), z_1) = 0$.

Hence, z_1 is a fixed point of f_1 in X. Also we can get that z_1 is a fixed point of f_2 in X. Therefore, f_1 and f_2 have a common fixed point which is $z_1 \in X$. Now, we need to prove the common fixed point z_1 is a unique. Let $z_2 \in X, z_2 \neq z_1$ be another fixed point of f_1 and f_2 such that $f_1(z_2) = z_2$ and $f_2(z_2) = z_2$, we get

$$\begin{aligned} d(z_1, z_2) = d\big(f_1(z_1), f_2(z_2)\big) &\leq m_1 d\big(z_1, f_1(z_1)\big) + m_2 d\big(z_2, f_2(z_2)\big) + m_3 d\big(z_1, f_2(z_2)\big) + \\ &\quad + m_4 d(z_1, z_2). \\ &= m_1 d(z_1, z_1) + m_2 d(z_2, z_2) + m_3 d(z_1, z_2) + m_4 d(z_1, z_2). \\ &= (m_3 + m_4) d(z_1, z_2) . \end{aligned}$$

So, $d(z_1, z_2) = 0$. Hence, z_1 is a unique common fixed point of f_1 and f_2.

Theorem 2.5 Let f_n be a self-mapping on a complete metric space X, with z_n fixed point for all $z_n \in X$, $\forall n$ respectively, such that

$$d\big(f_n(x), f_n(y)\big) \leq m_1 d\big(x, f_n(x)\big) + m_2 d\big(y, f_n(y)\big) + m_3 d\big(x, f_n(y)\big) + m_4 d(x, y) \qquad (2.13)$$

for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ such that $\sum_{i=1}^{4} m_i < 1$. If $f(z_1) = z_1$ and $f_n \to f$ then $f_n(z_1) = z_1$, $\forall n$.

Proof. We need to prove that $z_n \to z_1$, $\forall n$. Therefore,

$$\begin{aligned} d(z_n, z_1) = d(f_n(z_n), f(z_1)) &\leq m_1 d\big(z_n, f_n(z_n)\big) + m_2 d\big(z_1, f_1(z_1)\big) + m_3 d\big(z_n, f_1(z_1)\big) + \\ &\quad + m_4 d(z_n, z_1) \qquad (2.14) \\ &= (m_3 + m_4) d(z_n, z_1). \end{aligned}$$

So, $d(z_n, z_1) = 0$,

Therefore, $z_n = z_1$, $\forall n$.

Hence, z_1 is a unique fixed point of f_n on X.

Remark:

1. If $m_1 = m_2 = m_3 = m_4 = 0$, then $x = y$, but this is contradiction with Theorem 2.1.
2. If $m_1 = m_2 = m_3 = 0$, then Theorem 2.1 tends to S. Banach[1].
3. If $m_3 = m_4 = 0$, then Theorem 2.1 tends to R. Kanan[11].
4. If $m_3 = 0$, then Theorem 2.1 reduce to D. P. Shukla[14].
5. If we add $(m_5 d(y, f(x))$ such that $m_5 \in [0,1)$ and $\sum_{i=1}^{5} m_i < 1$, to the right side of (2.1), then Theorem 2.1 changes to G. E. Hardy[10].
6. If $m_1 = m_2 = m_4 = 0$, then 5. tends to B. Fisher[7].

3. APPLICATION

Gopal *et al*.[8] introduced the Banach's fixed point theorem as the following:

Definition 3.1 Let (X, d) be a metric and let $f: X \to X$ be a mapping

a) A point $x \in X$ is called a fixed point of f if $x = fx$.

b) f is called a contraction if there exists a fixed constant $\alpha < 1$ such that: $d(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in X$. (3.1)

Theorem 3.1 Let (X, d) be a complete metric space and $f: X \to X$ be a contraction, i.e. f satisfies (3.1) Then there exists a unique fixed point.

So, if we compare the condition (3.1) with the condition (2.1) we get that condition (3.1) implies that the continuality of the mapping is realized by[8] but (2.1) is not necessary. Therefore, (2.1) and (3.1) are completely independent.

Example 3.1 Let $X = [0,1]$ and $f(x_1) = \frac{x}{3}$, $x \in \left[0, \frac{1}{3}\right)$ $f(x_2) = \frac{x}{4}$, $x \in \left(\frac{1}{3}, 1\right]$, $f(x)$ is discontinuous at $x = \frac{1}{3}$. So (3.1) is not true since,

$d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2)$

$\Rightarrow \left|\frac{x}{3} - \frac{x}{4}\right| \leq \alpha |x_1, x_2|$

$\Rightarrow \left|\frac{x}{12}\right| \leq \alpha |x_1, x_2|$

Take $\alpha = \frac{1}{3}$ and $x_1 \in \left[0, \frac{1}{3}\right)$, $x_2 \in \left(\frac{1}{3}, 1\right]$ we get that $x \notin [0,1]$ $\forall x_1 \in \left[0, \frac{1}{3}\right)$, $x_2 \in \left(\frac{1}{3}, 1\right]$ and $\forall\, \alpha < 1$. On the other hand, the condition (2.1) is satisfied for all $m_1, m_2, m_3, m_4 \in (0,1)$ and $x_1 \in \left[0, \frac{1}{3}\right)$, $x_2 \in \left(\frac{1}{3}, 1\right]$. Hence, Theorem 2.1 is true with a unique fixed point which is $x = 0$, such that the fixed point of $f(x_1)$ and $f(x_2)$ need not have a common fixed point if $X = [0,1]$ by D. R. Smart[15].

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