JOURNAL OF COMPUTER AND MATHEMATICAL SCIENCES
An International Open Free Access, Peer Reviewed Research Journal
www.compmath-journal.org

ISSN 0976-5727 (Print)
ISSN 2319-8133 (Online)
Abbr:J.Comp.&Math.Sci.
2014, Vol.5(6): Pg.482-489

# Embedding of Petersen Graphs into Certain Trees

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(Received on: November 6, 2014)

## ABSTRACT

The study of graph embedding is an important topic in the theory of parallel computation. The existence of such an embedding demonstrates the ability of a parallel computer whose interconnection network is described by a guest graph. The Petersen graph is certainly one of the most famous objects that graph theorists have come across. In this paper, we present an algorithm for finding the exact wirelength of the Petersen graph $P(n, 1)$, i.e. the circular ladder into the $k$-rooted complete binary trees and binomial trees and prove its correctness using the Congestion lemma and Partition lemma.

**Keywords:** circular ladder, $k$-rooted complete binary tree, binomial trees, wirelength, edge congestion.

## 1. INTRODUCTION

Interconnection network plays a key role in the design of implementation of communication networks. A connection pattern of the component in a system is called an Interconnection Network or Network in short. Due to recent developments in parallel and distributed computing, the design of analysis of various interconnection network has been a main topic of research for the past few years. In developing a discriminant function for evaluating the 'goodness' of a network, in addition to three basic attributes - degree, diameter and node disjoint paths, a more complex attribute that needs to be considered is embeddability. The study of graph embedding ia an important topic in the theory of parallel computation. The existence of such an embedding demonstrates the ability of a parallel computer whose interconnection network is described by a guest graph. The concept of graph embedding has proven to be a successful one in understanding relationships between different architectures. The congestion sum or the wirelength of a graph embedding arises from the VLSI designs, data structures, data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on.

There are several results on the embedding problem of various architectures such as circular wirelength of generalized Petersen graphs[11], trees on cycles[5], trees on stars[22], hypercubes into grids[2], complete binary tree into grids[16], grids into grids[19], ladders and caterpillars into hypercubes[4], binary trees into hypercubes[6], complete binary trees into hypercubes[1], incomplete hypercube in books[7], *m*-sequencial *k*-ary trees into hypercubes[18], ternary tree into hypercube[9], enhanced and augmented hypercube into complete binary tree[13], and hypercubes into cylinders, snakes and caterpillars[14].

In this paper, we present an algorithm for finding the exact wirelength of circular ladders into the *k*-rooted complete binary trees and binomial trees and prove its correctness using the Congestion lemma[15] and Partition lemma[15].

## 2. PRELIMINARIES

**Definition 2.1.**[2] Let $G$ and $H$ be finite graphs with $n$ vertices. $V(G)$ and $V(H)$ denote the vertex sets of $G$ and $H$ respectively. $E(G)$ and $E(H)$ denote the edge sets of $G$ and $H$ respectively. An *embedding* $f$ of $G$ into $H$ is defined as follows:

(i) $f$ is a injective map from $V(G) \rightarrow V(H)$
(ii) $P_f$ is an injective map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in $H$ between $f(u)$ and $f(v)\}$.

The graph $G$ that is being embedded is called a *virtual graph* or a *guest graph* and $H$ is called a *host graph*. Some authors use the name *labeling* instead of embedding[1].

**Definition 2.2** [2] The *edge congestion* of an embedding $f$ of $G$ into $H$ is the maximum number of edges of the graph $G$ that are embedded on any single edge of $H$. Let $EC_f(G, H(e))$ denote the number of edges $(u, v)$ of $G$ such that $e$ is in the path $P_f(u, v)$ between $f(u)$ and $f(v)$ in $H$. In other words, $EC_f(G, H(e)) = \left|\{(u,v) \in E(G): e \in P_f(u,v)\}\right|$ where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in $H$ with respect to $f$.
The edge congestion problem of a graph $G$ into $H$ is to find an embedding of $G$ into $H$ that induces $EC(G, H)$.

**Definition 2.3.** [15] The *wirelength* of an embedding $f$ of $G$ into $H$ is given by

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H\big(f(u), f(v)\big) = \sum_{e \in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in $H$. Then the *wirelength* of $G$ into $H$ is defined as,

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all the embeddings.

The *edge isoperimetric problem*[3] is used to solve the wirelength problem when the host graph is a path and is NP-complete[8]. The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature[3].

***Problem* 1:** Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if

$\theta_G(m) = min_{A\subseteq V, |A|=m}|\theta_G(A)|$

where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $\theta_G(m) = |\theta_G(A)|$.

***Problem* 2:** Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given $m$, if $I_G(m) = max_{A\subseteq V, |A|=m}|I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u,v \in A\}$, then the problem is to find $A \subseteq V \ and\ |A| = m$ such that $I_G(m) = |I_G(A)|$.

For a given $m$, where $m$= 1, 2, ..., $|V|$, we consider the problem of finding a subset $A$ of vertices of $G$ such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called *optimal* with respect to Problems 1. We say that optimal subsets are *nested* if there exists a total order $O$ on the set $V$ such that for any $m$= 1, 2, ...,$n$, the collection of the first $m$ vertices in this order is an optimal subset. In this case, we call the order $O$ an optimal order[3].This implies that $WL(G, P_n) = \sum_{m=0}^{n} \theta_G(m)$, where $P_n$ is a path on $n$ vertices. Again, a subset $A$ of vertices of $G$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$ is said to be optimal with respect to Problems 2.

***Notation*:** $EC_f\ (G,H(e))$ will be represented by $EC_f\ (e)$. For any set $S$ of edges of $H$, $EC_f\ (S) = \sum_{e\in S} EC_f(e)$.

**Lemma 2.4.**(*Congestion lemma*) [15] Let $G$ be an $r$-regular graph and $f$ be an embedding of $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ leaves $H$ into 2 components $H_1$ and $H_2$ and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also $S$ satisfies the following conditions:

(i) For every edge $(a,b) \in G_i, i = 1,2$, $P_f\ (f\ (a), f\ (b))$ has no edges in $S$.

(ii) For every edge $(a,b)$ in $G$ with $a \in G_1$ and $b \in G_2$, $P_f\ (f\ (a), f\ (b))$ has exactly one edge in $S$.

(iii) $G_1$ is a maximum subgraph on $k$ vertices where $k = |V(G_1)|$.

Then $EC_f\ (S)$ is minimum and $EC_f\ (S) = rk - 2\,|E(G_1)|$.

**Lemma 2.5.**(*Partition lemma*) [15] Let $f : G \rightarrow H$ be an embedding. Let $\{S_1, S_2, ..., S_p\}$ be a partition of $E(H)$ such that each $S_i$ is an edge cut of $H$. Then,

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

## 3. PETERSEN GRAPHS

In 1950 a class of generalized Petersen graphs was introduced by Coxeter and around 1970 popularized by Frucht, Graver and Watkins. The Petersen graph is certainly one of the most famous objects that graph theorists have come across. This graph is a counter example to many conjectures: for example, it is not 1-factorizable despite being cubic and without bridges (Taits conjecture), and it is not hamiltonian. But being 3-transitive (that is, its automorphism group is transitive on directed paths of length 3), it is highly symmetric; however, it is not a Cayley graph! Many additional facts about the Petersen graph can be found in[12]. The Petersen graph appeared in the chemical literature as the graph that depicts a rearrangement of trigonal bipyramid complexes XY 5 with five different ligands when axial ligands become equatorial and equatorial ligands become axial.

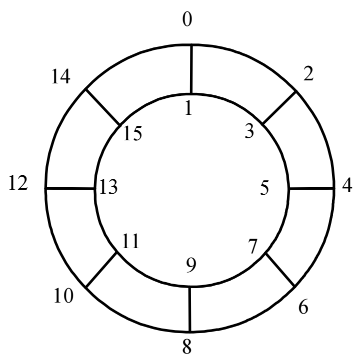


**Figure 1: Circular ladder *P*(8, 1)**

**Definition 3.1.** [24] The generalized Petersen graph $P(m, n)$, $1 \le m \le n-1$ and $n \neq 2m$, consists of an outer $n$-cycle $u_1, u_2, \ldots, u_n$, a set of $n$ spokes $(u_i, v_i)$, $1 \le i \le n$, and $n$ inner edges $(v_i, v_{i+m})$ with indices taken modulo $n$. It is a 3-regular graph and contains $2n$ vertices and $3n$ edges. See Figure 1.

**Parallel labeling** [11]. For $1 \le i \le n$, we call the vertices $u_i$ and $v_i$ of $P(m, n)$ as outer rim and inner rim vertices respectively and label the vertices $u_i$ and $v_i$ as $2i-2$ and $2i-1$ respectively. We call this labeling as parallel labeling of the generalized Petersen graph $P(m, n)$.

We know that the generalized Petersen graph $P(n, 1)$, $n \ge 3$ is the circular ladder $K_2 \times C_n$.

In this paper we consider the generalized Petersen graph $P(n, 1)$, $n \ge 3$ for our discussion.

**Theorem 3.2.** [11] The number of edges in a subgraph induced by any set of $k$ vertices of $P(n, 1)$, $3 \le k \le n$ is atmost $k + \lfloor k/2 \rfloor - 2$ for $n > 3$ .

**Theorem 3.3.** [11] Let $H$ be a subgraph of $P(n, 1)$ induced by $k$ vertices, $3 \le k \le n$ such that,

(i) if $k$ is even, the labels of the $k$ vertices are $\{i+1, i+2, \ldots, i+k\}$ and

(ii) if $k$ is odd, the labels of $k-1$ vertices are $\{i+1, i+2, \ldots, i+k-1\}$ and the $k^{th}$ vertex is the vertex labeled $i-1$, $i$, $i+k$, or $i+k+1$ where $i$ is odd and the labels are taken modulo $2n$. Then $H$ is a maximum subgraph of $P(n, 1)$, $n \ge 3$.

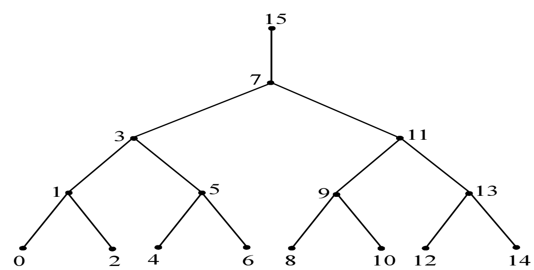


**Figure 2: 1- rooted complete binary tree $T_n^1$**

## 4. WIRELENGTH OF CIRCULAR LADDERS INTO THE k-ROOTED COMPLETE BINARY TREES

Complete binary trees are perfectly balanced and have the maximum possible number of nodes, given their height. However, they exist only when $n$ is one less than a power of 2. For any non-negative integer $n$, the complete binary tree of height $n$, denoted by $T_n$, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Thus a complete binary tree $T_n$ has $n$ levels and level $i$, $1 \leq i \leq n$, contains $2^i - 1$ vertices. Thus $T_n$ has exactly $2^n - 1$ vertices.

**Definition 4.1.** [10] The 1-rooted complete binary tree $T_n^1$ is obtained from a complete binary tree $T_n$ by attaching to its root a pendant edge. The new vertex is called the root of $T_n^1$ and is considered to be at level 0. The $k$-rooted complete binary tree $T_n^k$ is obtained by taking $k$ vertex disjoint 1-rooted complete binary trees $T_n^1$ on $2^n$ vertices with roots say $r_1, r_2, \ldots, r_k$ and adding the edges $(r_i, r_i+1)$, $1 \leq i \leq k-1$. See Figure 2(a).

**Definition 4.2.** [26] Let $T$ be a tree having root $r$ with sons $v_1, v_2, \ldots, v_k$, $k \geq 0$. In the case $k = 0$, the tree consists of a single vertex $r$. For a binary tree, an inorder traversal is defined recursively as follows: (i) Visit in inorder the left subtree of the root $r$ (if it exists). (ii) Visit the root $r$. (iii) Visit in inorder the right subtree of $r$ (if it exists).

**Embedding Algorithm A**

**Input :** A generalized Petersen graph $P(2^{n-1}, 1)$ and a 1-rooted complete binary tree $T_n^1$ where $n > 3$.

**Algorithm :** Label the vertices of $P(2^{n-1}, 1)$ using parallel labeling. Label the vertices of $T_n^1$ as 0, 1, ..., $n - 1$ using inorder labeling.

**Output :** An embedding $f$ of $P(2^{n-1}, 1)$ into $T_n^1$ given by $f(x) = x$ with minimum wirelength.

**Proof of correctness :** For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2j-1}$ be the cut edge of the 1-rooted complete binary tree $T_n^1$, which has one vertex in level $n - j$ and the other vertex in level $n - j + 1$, such that $S_i^{2j-1}$ disconnects $T_n^1$ into two components $X_i^{2j-1}$ and $\bar{X}_i^{2j-1}$ where $V(X_i^{2j-1})$ is $\{2^j(i-1), 2^j(i-1)+1, 2^j(i-1)+2, \cdots, 2^j(i-1)+(2^j-2)\}$. Let $G_i^{2j-1}$ and $\bar{G}_i^{2j-1}$ be the inverse images of $X_i^{2j-1}$ and $\bar{X}_i^{2j-1}$ under $f$ respectively. By Theorem 3.3 (ii), $G_i^{2j-1}$ is an optimal set in $P(2^{n-1}, 1)$. Thus the cut edge $S_i^{2j-1}$ satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore, $EC_f(S_i^{2j-1})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$. The Partition Lemma implies that the wirelength is minimum.

**Theorem 4.3.** The exact wirelength of a generalized Petersen graph $P(2^{n-1}, 1)$ into the 1-rooted complete binary tree $T_n^1$ is given by,

$$WL(P(2^{n-1}, 1), T_n^1) = 27(2^{n-3}) + 3 + \sum_{j=4}^{n-1} 2^{n-j}\left[3(2^j - 1) - 2\left(2^j + \left\lfloor \frac{2j-1}{2} \right\rfloor - 3\right)\right]$$

**Embedding Algorithm B**

**Input :** A generalized Petersen graph $P(2^{n-1}, 1)$ and a $k$-rooted complete binary tree $T_{n_1}^k$, where $n \geq n_1$, $n, n_1 > 3$, $k = 2^{n-n_1}$.

**Algorithm :** Label the vertices of $P(2^{n-1}, 1)$ using parallel labeling. Label the vertices of $T_{n_1}^k$, $n \geq n_1$, $n, n_1 > 3$, $k = 2^{n-n_1}$ as follows: Let $T_{n_1}^{1,1}, T_{n_1}^{1,2}, \ldots, T_{n_1}^{1,k}$ be the $k$ vertex disjoint 1-rooted complete binary trees of $T_n^1$. Label the vertices of $T_{n_1}^{1,i}$, $1 \leq i \leq k$, using inorder labeling from $(i-1)2^{n_1}$ to $i2^{n_1} - 1$.

**Output :** An embedding $f$ of $P(2^{n-1}, 1)$ into $T_{n_1}^k$ given by $f(x) = x$ with minimum wirelength.

**Proof of correctness :** For $i = 1, 2, \ldots, 2^{n-n_1}$, let $S_i^{2^{n_1}}$ be the cut edge of the $k$-rooted omplete binary tree $T_{n_1}^k$, which has one vertex in level $n$ - $j$ and the other vertex in level $n$ - $j$ + 1, such that $S_i^{2^{n_1}}$ disconnects $T_{n_1}^k$ into two components $X_i^{2^{n_1}}$ and $\bar{X}_i^{2^{n_1}}$ where $V(X_i^{2^{n_1}})$ is $\{2^{n_1}(i-1), 2^{n_1}(i-1) + 1, 2^{n_1}(i-1) + 2, \cdots, 2^{n_1}(i-1) + (2^{n_1} - 1)\}$. Let $G_i^{2^{n_1}}$ and $\bar{G}_i^{2^{n_1}}$ be the inverse images of $X_i^{2^{n_1}}$ and $\bar{X}_i^{2^{n_1}}$ under $f$ respectively. By Theorem 3.3 (ii), $G_i^{2^{n_1}}$ is an optimal set in $P(2^{n-1}, 1)$. Thus the cut edge $S_i^{2^{n_1}}$ satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore, $EC_f(S_i^{2^{n_1}})$ is minimum for $i = 1, 2, \ldots, 2^{n-n_1}$. The Partition Lemma implies that the wirelength is minimum.

By the proof of the Embedding Algorithm A and the discussion above, it is sufficient to prove that the cut edge $(r_i, r_i+1)$, $1 \leq i \leq k-1$, where $r_i$ is the root of $T_{n_1}^{1,1}$, $1 \leq i \leq k$, has minimum edge congestion. The cut edge $(r_i, r_i+1)$, $1 \leq i \leq k-1$, of $T_{n_1}^k$, disconnects $T_{n_1}^k$ into two components $X_i$ and $\bar{X}_i$ where $V(X_i) = \{0, 1, \cdots, i2^{n_1} - 1\}$. Let $G_i$ and $\bar{G}_i$ be the inverse images of $X_i$ and $\bar{X}_i$ under $f$ respectively. By Theorem 3.3 (ii), $G_i$ is an optimal set in $P(2^{n-1}, 1)$. Thus the cut edge $(r_i, r_i+1)$, $1 \leq i \leq k-1$, satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore, $EC_f((r_i, r_i+1))$ is minimum for $i = 1, 2, \ldots, k-1$. The Partition Lemma implies that the wirelength is minimum.

**Theorem 4.4.** The exact wirelength of a generalized Petersen graph $P(2^{n-1}, 1)$ into the k-rooted complete binary tree $T_{n_1}^k$ is given by,

$$WL(P(2^{n-1}, 1), T_{n_1}^k) = 27(2^{n-3}) + 3 + \sum_{j=4}^{n-1} 2^{n-j}\left[3(2^j - 1) - 2\left(2^j + \left\lfloor \frac{2^j-1}{2} \right\rfloor - 3\right)\right] + 4(k-1).$$

## 5. WIRELENGTH OF CIRCULAR LADDERS INTO BINOMIAL TREES

**Definition 5.1.** [10] A binomial tree $B_0$ of height 0 is a single vertex. For all $n > 0$, a binomial tree $B_n$ of height $n$ is a tree formed by joining the roots of two binomial trees of height $n - 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height $n$ has $2^n$ vertices.

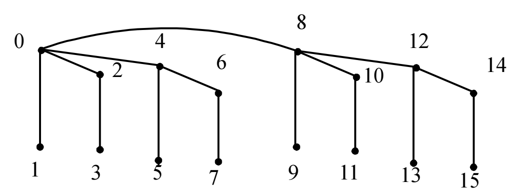


**Figure 3: Binomial Tree $B_4$**

### Embedding Algorithm C

**Input :** A generalized Petersen graph $P(2^{n-1}, 1)$ and the binomial tree $B_n$.

**Algorithm :** Label the vertices of $P(2^{n-1}, 1)$ using parallel labeling. Label the vertices of $B_n$ as 0, 1, ..., $n-1$ as shown in Figure 3.

**Output :** An embedding $f$ of $P(2^{n-1}, 1)$ into $B_n$ given by $f(x) = x$ with minimum wirelength.

**Proof of correctness :** For $j = 1, 2, \ldots, n$ and $i = 1, 2, \cdots, 2^{n-j}$, let $S_i^{2^j-1} = \{(2^{j-1}(2i-1), 2^{j-1}(2i-2))\}$ be the cut edge of $B_n$ such that $S_i^{2^j-1}$ disconnects $B_n$ into two components $X_i^{2j-1}$ and $\bar{X}_i^{2j-1}$ where $V(X_i^{2j-1})$ is $\{2^{j-1}(2i-1), 2^{j-1}(2i-1)+1, 2^{j-1}(2i-1)+2, \cdots, 2^{j-1}(2i-1)+(2^{j-1}-1)\}$. Let $G_i^{2j-1}$ and $\bar{G}_i^{2j-1}$ be the inverse images of $X_i^{2j-1}$ and $\bar{X}_i^{2j-1}$ under $f$ respectively. By Theorem 3.3 (ii), $G_i^{2j-1}$ is an optimal set in $P(2^{n-1}, 1)$. Thus the cut edge $S_i^{2j-1}$ satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore, $EC_f(S_i^{2j-1})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$. The Partition Lemma implies that the wirelength is minimum.

**Theorem 5.2.** The exact wirelength of a generalized Petersen graph $P(2^{n-1}, 1)$ into the binomial tree $B_n$ is given by,

$WL(P(2^{n-1}, 1), B_n) = 3(2^{n-1}) + 4 + \sum_{j=2}^{n-1} 2^{n-j}[3(2^{j-1}) - 2^j - 2^{j-1} + 4]$

## 6. CONCLUSION

In this paper, we have found the exact wirelength of the circular ladder into certain types of trees. Finding the exact wirelength of variety of such trees by varying the guest graph would be of great interest.

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