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Refinement Invariants of Binary Images

Jack Weinstein

Department of Mathematics,
University of Haifa, Mount Carmel, Haifa, ISRAEL.
email: jack-weinstein@hotmail.com.

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ABSTRACT

Some uniqueness results are proven for the Euler numbers of binary digital images.

Keywords: Euler number; binary digital image; refinement invariant.

1. BASICS

The spaces we are dealing with are *binary digital images*. We define a binary digital image as a function $P:\mathbf{Z}^2 \to \{0,1\}$. A coordinate system in $\mathbf{Z}^2$ is chosen such that the first axis points downward (the *row* axis) and the second axis points to the right (the *column* axis). As usual, an element $(i, j) \in \mathbf{Z}^2$ can be regarded as a point (placed at row i and column j), or as a square placed with its center or with its upper-left corner at coordinates (i, j); such an element is usually called a *pixel*. If $P(i, j) = 0$, the pixel (i, j) is called a *background* point; otherwise, if $P(i, j) = 1$, the pixel (i, j) is called a *foreground* point. We will assume in this paper that background pixels (0-pixels) are black and foreground pixels (1-pixels) are white. We also assume the number of 1-pixels to be finite; we can therefore restrict each image to a digital rectangle.

Given a binary digital image P, the set of white pixels defines a graph $gr(P)$. The vertices of this graph are single pixels, represented as $\begin{bmatrix}1\end{bmatrix}$; its faces are quadruples of pixels, represented as $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$; and its edges are horizontal pairs $\begin{bmatrix}1 & 1\end{bmatrix}$, vertical pairs $\begin{bmatrix}1\\ 1\end{bmatrix}$, main diagonal pairs $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, and secondary diagonal pairs $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. At some point in the paper we will use the term *bit-quad* for all quadruples of pixels represented as $\begin{bmatrix}X & X\\ X & X\end{bmatrix}$ (where each X denotes either of the values 0 or 1).

2. RESOLUTION REFINEMENTS

Let P be a binary digital image. The *resolution refinement* of P is a binary digital image Q, obtained from P by dividing each pixel into four smaller ones; this division is performed by cutting each square by a horizontal line and a vertical line, both passing through its center.

A convenient way to represent the resolution refinement of a binary image is to place the pixels of the original image P with their corners at integral coordinates, then multiply all coordinates by 2 and perform the subdivision (and finally move back the pixels with their centers at integral coordinates, if desired).

A function ρ on the class of binary digital images will be called a *resolution refinement invariant* (or just a *refinement invariant*), if $\rho(Q)=\rho(P)$ whenever Q is the resolution refinement of P. The aim of this paper is to characterize all linear, real valued, resolution refinement invariants of certain kinds.

In the following we will denote by $\#(P;\pi)$ the number of occurrences of the local pattern π in the graph $gr(P)$.

3. TECHNICAL DETAILS

Consider the function

$$\rho(P)=\alpha_1\cdot\#(P;[1])+\alpha_3\cdot\#(P;[1\ \ 1])+\alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

where the α_i's ($i=1,3,5,15$) are real numbers.

If Q is a resolution refinement of P, then

- each vertex $[1]$ of P generates in Q: four vertices $[1]$, two horizontal edges $[1\ \ 1]$, two vertical edges $\begin{bmatrix}1\\1\end{bmatrix}$, and one square $\begin{bmatrix}1&1\\1&1\end{bmatrix}$;
- each horizontal edge $[1\ \ 1]$ of P generates in Q, in excess: two horizontal edges $[1\ \ 1]$, and one square $\begin{bmatrix}1&1\\1&1\end{bmatrix}$;
- each vertical edge $\begin{bmatrix}1\\1\end{bmatrix}$ of P generates in Q, in excess: two vertical edges $\begin{bmatrix}1\\1\end{bmatrix}$, and one square $\begin{bmatrix}1&1\\1&1\end{bmatrix}$;
- each square face $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ of P generates in Q, in excess: one square $\begin{bmatrix}1&1\\1&1\end{bmatrix}$.

Therefore,

$$\#(Q;[1]) = 4\cdot\#(P;[1])$$
$$\#(Q;[1\ \ 1]) = 2\cdot\#(P;[1]) + 2\cdot(P;[1\ \ 1])$$
$$\#\left(Q;\begin{bmatrix}1\\1\end{bmatrix}\right) = 2\cdot\#(P;[1]) + 2\cdot\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)$$
$$\#\left(Q;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) = \#(P;[1]) + (P;[1\ \ 1]) + \left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

and we get

$$\begin{aligned}\rho(Q) &= \alpha_1\cdot\#(Q;[1]) + \alpha_3\cdot\#(Q;[1\ \ 1]) + \alpha_5\cdot\#\left(Q;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{15}\cdot\#\left(Q;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\\ &= (4\alpha_1 + 2\alpha_3 + 2\alpha_5 + \alpha_{15})\cdot\#(P;[1]) + (2\alpha_3 + \alpha_{15})\cdot\#(P;[1\ \ 1])\\ &\qquad + (2\alpha_5 + \alpha_{15})\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\end{aligned}$$

In the sequel we will need the difference

$$\begin{aligned}\rho(Q) - \rho(P) &= (3\alpha_1 + 2\alpha_3 + 2\alpha_5 + \alpha_{15})\cdot\#(P;[1])\\ &\qquad + (\alpha_3 + \alpha_{15})\cdot\#(P;[1\ \ 1]) + (\alpha_5 + \alpha_{15})\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)\end{aligned}$$

If $\rho(Q) = \rho(P)$, then for each binary digital image P,

$$(D)\quad (3\alpha_1 + 2\alpha_3 + 2\alpha_5 + \alpha_{15})\cdot\#(P;[1]) + (\alpha_3 + \alpha_{15})\cdot\#(P;[1\ \ 1]) + (\alpha_5 + \alpha_{15})\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) = 0$$

If we extend the function ρ to one of the following three functions

$$\rho(P) = \alpha_1\cdot\#(P;[1]) + \alpha_3\cdot\#(P;[1\ \ 1]) + \alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) + \alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$
$$\rho(P) = \alpha_1\cdot\#(P;[1]) + \alpha_3\cdot\#(P;[1\ \ 1]) + \alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) + \alpha_6\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

and

$$\begin{aligned}\rho(P) &= \alpha_1\cdot\#(P;[1]) + \alpha_3\cdot\#(P;[1\ \ 1]) + \alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\\ &\qquad + \alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) + \alpha_6\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\end{aligned},$$

then we easily see that each of the bit-quads $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ in P generates just one bit-quad of the same kind in Q; therefore, there will be no change in (D).

4. EULER NUMBERS

We consider the following well known, linear, integer valued functions on the class of binary digital images:

the 4-connectivity Euler number

$$\chi^{(4)}(P)=\#(P;[1])-\#(P;[1\ 1])-\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

the main diagonal 6-connectivity Euler number

$$\chi^{(6m)}(P)=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$=\#(P;[1])-\#(P;[1\ 1])-\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

the secondary diagonal 6-connectivity Euler number

$$\chi^{(6s)}(P)=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$=\#(P;[1])-\#(P;[1\ 1])-\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

and the 8-connectivity Euler number

$$\chi^{(8)}(P)=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$=\#(P;[1])-\#(P;[1\ 1])-\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

For the 4-connectivity and 8-connectivity Euler numbers, see the references [G71], [RK76, p.349], [RK82, pp.245-247]; for the two 6-connectivity Euler numbers see the references [G71], [H86, pp.76-77 and p.85].

Each of these Euler number functions is a refinement invariant. For a proof, it is enough to replace in (D)

$$\alpha_1=\alpha_{15}=1,\quad \alpha_3=\alpha_5=-1,$$

and we get $\rho(Q)=\rho(P)$. However, these four Euler numbers are not independent, since

$$\chi^{(4)}(P)+\chi^{(8)}(P)=\chi^{(6m)}(P)+\chi^{(6s)}(P)$$

On the other hand, every three of them are independent. Assume, to the contrary, that

$$\alpha\cdot\chi^{(4)}(P)+\beta\cdot\chi^{(6m)}(P)+\gamma\cdot\chi^{(6s)}(P)=0$$

for some constants $\alpha,\beta,\gamma\in R$. Then

$$(\alpha+\beta+\gamma)\cdot\chi^{(4)}(P)-\beta\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\gamma\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=0$$

for all images P .

By choosing the image $P=[1]$ (only one white pixel in the image), we get $\chi^{(4)}(P)=1$, $\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=0$, $\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=0$; therefore, $\alpha+\beta+\gamma=0$.

By choosing the image $P=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (only two white pixels in the image, forming a main diagonal edge), we get $\chi^{(4)}(P)=2$, $\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=1$, $\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=0$; therefore, $2(\alpha+\beta+\gamma)=\beta$.

By choosing the image $P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (only two white pixels in the image, forming a secondary diagonal edge), we get $\chi^{(4)}(P)=2$, $\#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)=0$, $\#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)=1$; therefore, $2(\alpha+\beta+\gamma)=\gamma$.
The solution of this equation system is $\alpha=\beta=\gamma=0$.
The independence of the other triples follows similarly.

The aim of this paper is to prove some uniqueness results, according to which these four Euler number functions are, essentially, the only linear, real valued refinement invariants on the class of binary digital images.

5. UNIQUENESS THEOREMS

Theorem 1. Every refinement invariant of the form

$$\rho(P)=\alpha_1\cdot\#(P;[1])+\alpha_3\cdot\#(P;[1\ \ 1])+\alpha_5\cdot\#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)+\alpha_{15}\cdot\#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\alpha_1,\alpha_3,\alpha_5,\alpha_{15}\in R$$

can be expressed in a unique way as a linear combination

$$\rho(P)=\alpha\cdot\chi^{(4)}(P),\quad \alpha\in R\ .$$

Proof: We use identity (D), since ρ is a refinement invariant.

By choosing the image $P=[1]$ (only one white pixel in the image), we get $\#(P;[1])=1$, $\#(P;[1\ \ 1])=0$, $\#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=0$; therefore, $3\alpha_1+2\alpha_3+2\alpha_5+\alpha_{15}=0$.

By choosing the image $P=[1\ \ 1]$ (only two white pixels in the image, forming a horizontal edge), we get $\#(P;[1])=2$, $\#(P;[1\ \ 1])=1$, $\#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=0$; therefore, $6\alpha_1+5\alpha_3+4\alpha_5+3\alpha_{15}=0$.

By choosing the image $P=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (only two white pixels in the image, forming a vertical edge), we get $\#(P;[1])=2$, $\#(P;[1\ \ 1])=0$, $\#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=1$; therefore, $6\alpha_1+4\alpha_3+5\alpha_5+3\alpha_{15}=0$.

We derived the following system of equations

$$3\alpha_1+2\alpha_3+2\alpha_5+\alpha_{15}=0$$
$$\alpha_3+\alpha_{15}=0$$
$$\alpha_5+\alpha_{15}=0$$

and from these it turns out that

$\alpha_1=\alpha_{15}=\alpha$, $\alpha_3=\alpha_5=-\alpha$, for some $\alpha\in R$. Therefore, $\rho(P)=\alpha\cdot\chi^{(4)}(P)$.

Theorem 2. Every refinement invariant of the form

$$\rho(P)=\alpha_1\cdot\#(P;[1])+\alpha_3\cdot\#(P;[1\ \ 1])+\alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)+\alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$\alpha_1,\alpha_3,\alpha_5,\alpha_{15},\alpha_9\in R$$

can be expressed in a unique way as a linear combination

$$\rho(P)=\alpha\cdot\chi^{(4)}(P)+\delta\cdot\chi^{(6m)}(P),\quad \alpha,\delta\in R\ .$$

Proof: By not being involved in identity (D), α_9 is an independent parameter. As in the proof of Theorem 1, we get $\alpha_1=\alpha_{15}=\beta$, $\alpha_3=\alpha_5=-\beta$, for some $\beta\in R$. Therefore,

$$\rho(P)=\beta\cdot\chi^{(4)}(P)+\alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),$$

or

$$\rho(P)=\alpha\cdot\chi^{(4)}(P)+\delta\cdot\chi^{(6m)}(P)$$

where $\alpha=\beta+\alpha_9$, $\delta=-\alpha_9$.

Theorem 3. Every refinement invariant of the form

$$\rho(P)=\alpha_1\cdot\#(P;[1])+\alpha_3\cdot\#(P;[1\ \ 1])+\alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)+\alpha_6\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$\alpha_1,\alpha_3,\alpha_5,\alpha_{15},\alpha_6\in R$$

can be expressed in a unique way as a linear combination

$$\rho(P)=\alpha\cdot\chi^{(4)}(P)+\delta\cdot\chi^{(6s)}(P),\quad \alpha,\delta\in R\ .$$

Proof: Similar to that of Theorem 2.

Theorem 4. Every refinement invariant of the form

$$\rho(P)=\alpha_1\cdot\#(P;[1])+\alpha_3\cdot\#(P;[1\ \ 1])+\alpha_5\cdot\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)+\alpha_{15}\cdot\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$
$$+\alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\alpha_6\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$\alpha_1,\alpha_3,\alpha_5,\alpha_{15},\alpha_9,\alpha_6\in R$$

can be expressed in a unique way as a linear combination

$$\rho(P)=\alpha\cdot\chi^{(4)}(P)+\delta_m\cdot\chi^{(6m)}(P)+\delta_s\cdot\chi^{(6s)}(P),\quad \alpha,\delta_m,\delta_s\in R\ .$$

Proof: As in the proofs of Theorems 2 and 3, we get $\alpha_1=\alpha_{15}=\beta$, $\alpha_3=\alpha_5=-\beta$, for some $\beta\in R$, and α_9, α_6 are independent parameters. Therefore,

$$\rho(P)=\beta\cdot\chi^{(4)}(P)+\alpha_9\cdot\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\alpha_6\cdot\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

or

$$\rho(P)=\alpha\cdot\chi^{(4)}(P)+\delta_m\cdot\chi^{(6m)}(P)+\delta_s\cdot\chi^{(6s)}(P)$$

where

$$\alpha=\beta+\alpha_9+\alpha_6,\quad \delta_m=-\alpha_9,\quad \delta_s=-\alpha_6.$$

6. THE BIT-QUAD REPRESENTATION

We can represent each of the smaller digital patterns whose counts appear in the Euler number definition, as a bit-quad (2×2 digital square) containing it in its right-bottom corner; and thus we get

$$\begin{aligned}\#\left(P;[1]\right) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\ &\quad+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\end{aligned}$$

$$\#\left(P;[1\ \ 1]\right)=\#\left(P;\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

$$\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)=\#\left(P;\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

These lead us to the following identities

$$\chi^{(4)}(P)=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$\chi^{(6m)}(P)=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)$$

$$\begin{aligned}\chi^{(6s)}(P)&=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\end{aligned}$$

$$\begin{aligned}\chi^{(8)}(P)&=\chi^{(4)}(P)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\ &=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\end{aligned}$$

When computing an Euler number for some binary digital image, we have to restrict the image domain to some rectangle containing all white pixels; however, we have to make sure for our images that one of the rows (first or last) is completely black and that one of the columns (first or last) is completely black.

For each of these four identities, we have three more variants, according to the direction of the corner into which the small patterns are placed in the bit-quads:

$$\left(E^{(4)}\right)\qquad \begin{aligned}\chi^{(4)}(P)&=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\\ &=\#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &=\#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\\ &=\#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\end{aligned}$$

$$\left(E^{(6m)}\right)\qquad \begin{aligned}\chi^{(6m)}(P) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\end{aligned}$$

$$\left(E^{(6s)}\right)\qquad \begin{aligned}\chi^{(6s)}(P) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)\end{aligned}$$

$$\left(E^{(8)}\right)\qquad \begin{aligned}\chi^{(8)}(P) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\end{aligned}$$

We also can derive a compact representation for all four Euler numbers:

$$\left(C^{(4)}\right)\qquad \begin{aligned}\chi^{(4)}(P) &= \#\left(P;\begin{bmatrix}X&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&1\\X&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}1&0\\0&X\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&X\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)\end{aligned}$$

$$\left(C^{(6m)}\right)\qquad \begin{aligned}\chi^{(6m)}(P) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&1\\X&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&X\\0&1\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)\\ &= \#\left(P;\begin{bmatrix}0&X\\1&0\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}1&0\\X&1\end{bmatrix}\right)\end{aligned}$$

$$\left(C^{(6s)}\right) \qquad \chi^{(6s)}(P) = \#\left(P;\begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & X \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & X \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} X & 1 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)$$
$$\left(C^{(8)}\right) \qquad \chi^{(8)}(P) = \#\left(P;\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & X \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} X & 1 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \#\left(P;\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix}\right)$$

We have thus a variety of ways to compute the Euler numbers in a more efficient manner.

Some of these formulas are well known; however, a complete list does not seem to have been published anywhere.

7. REFERENCES

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