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Best Proximity Point Theorems in Metric Spaces

J. Beny[1*], J. Maria Joseph[2] and M. Marudai[3]

[1]PG and Research Department of Mathematics,
Holy Cross College (Autonomous), Trichy-620 002, INDIA.
[2]PG and Research Department of Mathematics,
St.Joseph College (Autonomous), Trichy-620 002, INDIA.
[3]School of Mathematical Sciences,
Bharathidasan University, Trichy, INDIA.
*Corresponding Author: benykutty@gmail.com.

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ABSTRACT

In this paper, we prove existence and convergence of best proximity point theorems in metric spaces. Our results generalize and unify some results in the recent literature.

Keywords: Fixed point, complete metrics space, best proximity point, contraction.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays a vital role in Mathematical analysis. Best approximations and best proximity points are considered as an extension of fixed point theory. In 1922, Stefan Banach has come up with beautiful theorem known as banach contraction theorem. This theorem laid foundation for all fixed point theorems. Eldred and Veeramani[2] proved existence and convergence of best proximity points in 2006. Then, many authors presented best proximity point results for different types of mappings[1,3,5,6]. In this section, we provide some definitions.

Defintion1.1. [1] A subset k of a metric space X is boundedly compact if each bounded sequence in k has subsequence converging to a point in k . Suppose X is a uniformly convex (and hence reflexive) Banach space with modules convexity δ .Then $\delta(\varepsilon) > 0$ $for \varepsilon > 0$ and $\delta(.)$ is strictly increasing. Moreover, if $x, y, p \in X, R > 0$ and $r \in 0, 2R$,

$$\left.\begin{array}{r}\|x-p\|\leq R\\ \|y-p\|\leq R\\ \|x-y\|\geq r\end{array}\right\}\Rightarrow\left\|\frac{x+y}{2}-p\right\|\leq\left(1-\delta\left(\frac{r}{R}\right)\right)R$$

Definition 1.2. [4] Let X be a nonempty set . A function $d:X\times X\to R^{+}$ is called a metric provide that ,for all $x,y,z\in X$,

1. $d(x,y)=0$ if and only if $x=y$
2. $d(x,y)=d(y,x)$
3. $d(x,z)\leq d(x,y)+d(y,z)$.

A pair (X,d) is called a metric space.

Definition 1.3.[1] Let (X,d) be a metric space $\{x_n\}$ be a sequence in X and $x\in X$. Then,

a). The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x ,if for every $\varepsilon>0$ there exists a $n_0\in\mathbb{N}$ such that $d(x_n,x)<\varepsilon$ for all $n>n_0$ and this fact is represented by $\lim_{n\to\infty}=x$ or $x_n\to x$ as $n\to\infty$.

b). The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if for every $\varepsilon>0$ there exits $n_0\in\mathbb{N}$ such that $d(x_n,x_{n+p})<\varepsilon$ for all $n<n_0,\ p>0$ or equivalently ,if $\lim_{n\to\infty}(x_n,x_{n+p})=0$ for all p.

c). (X,d) is said to be a complete metric space if every Cauchy sequence in X converges to some $x\in X$.

Definition 1.4.[2] The set B is said to be approximatively compact with respect to A if every sequence sequence $\{y_n\}$ of B satisfying the condition that $d(x,y_n)\to d(x,B)$ for some $x\in A$ has a convergent subsequence.
In this setting, we recall the following notions:

$$dist(A,B)=\inf\{d(a,b)\,/\,a\in A,b\in B\}$$

$$A_0=\{a\in A/d(a,b)=dist(A,B)\ for some b\in B\}$$

$$B_0=\{b\in B/d(a,b)=dist(A,B)\ for some a\in A\}$$

2. MAIN RESULTS

Definition 2.1 Let A and B be nonempty subsets of a complete metric space (X, d) A cyclic map $F : A \cup B \to A \cup B$ is said to be generalized best proximity contraction mapping if there exists $l_i \geq 0, i = 1, 2, 3,$ such that

$$d(Fa, Fb) \leq l_1 d(a,b) + l_2 d(a,B) + l_3 d(A,b) + (1-(l_1+l_2+l_3))dist(A,B), \qquad (2.1)$$

Where for all $a \in A, b \in B$ and $(l_1 + l_2 + l_3) < 1$.

Example 2.2 Let $X = R$. Define $d : X \times X \to [0, \infty)$ by $d(a,b) = |a-b|$, for all $a, b \in X$. Then (X, d) is a complete metric space.

Define $F : X \to X$ by $Fa = \frac{-a}{5,}$ for all $a, b \in X$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$.

Let $a \in A$ then $Fa \in B.$ Let $b \in B$ then $Fb \in A.$ Therefore, $F(A) \subset B$ and $F(B) \subset A.$ Then F is a cyclic map.

$$d(Fa, Fb) = \frac{1}{5} d(a,b)$$

$$\leq l_1 d(a,b) + l_2 d(a,B) + l_3 d(A,b) + \pi 1 - (l_1 + l_2 + l_3)\ dist(A,B),$$

$$\left[where, l_1 = \frac{1}{5}, l_2 = l_3 = 0, dist(A,B) = 0 \right]$$

Hence, F is generalized best proximity contraction mapping.

Theorem 2.3 Let A and B be non empty subsets of a metric space X. Let $F : A \cup B \to A \cup B$ be generalized best proximity contraction mapping. Then $d(a_n, Fa_n)$ converges to dist (A, B), where $a_{n+1} = Fa_n, n \in \mathbb{N} \cup \{0\}$.

Proof:

$$d(a_n, a_{n+1}) = d(Fa_{n-1}, Fa_n)$$

$$\leq l_1 d(a_{n-1}, a_n) + l_2 d(a_{n-1}, B) + l_3 d(a_n, A) + (1 + (l_1 + l_2 + l_3))\ dist(A,B)$$

$$\leq l_1 d(a_{n-1}, a_n) + l_2 d(a_{n-1}, a_n) + l_3 d(a_{n-1}, a_n) + (1 - (l_1 + l_2 + l_3))\ dist(A,B)$$

$$= (l_1 + l_2 + l_3)\ d(a_{n-1}, a_n) + (1 - (l_1 + l_2 + l_3))\ dist(A,B)$$

$$= (l_1 + l_2 + l_3)\ d(T_{a_{n-2}}, T_{a_{n-1}}) + (1 - (l_1 + l_2 + l_3))\ dist(A,B)$$

$$\leq (l_1+l_2+l_3)\,(l_1 d(a_{n-2},a_{n-1}) + l_2 d(a_{n-2},B) + l_3 d(a_{n-1},A)) + (1-(l_1+l_2+l_3))\,dist(\mathrm{A,B})$$

$$+ (1-(l_1+l_2+l_3))\,dist(\mathrm{A,B})$$

$$\leq (l_1+l_2+l_3)\,(l_1 d(a_{n-2},a_{n-1}) + l_2 d(a_{n-2},a_{n-1}) + l_3 d(a_{n-1},a_{n-2})) + (1-(l_1+l_2+l_3))\,\mathrm{dist(A,B)}$$

$$+ (1-(l_1+l_2+l_3))\,dist(A,B)$$

$$= (l_1+l_2+l_3)^2 d(a_{n-2},a_{n-1}) + (1-(l_1+l_2+l_3))^2\, dist(A,B)$$

Inductively ,We obtain

$$d(a_n,a_{n+1}) \leq (l_1+l_2+l_3)^n d(a_0,a_1) + (1-(l_1+l_2+l_3))^n\, dist(A,B).$$

Hence, $d(a_n,a_{n+1}) \to dist(A,B)$

Theorem 2.4 Let A and B be non empty subsets of metric space X. Let $F: A\cup B \to A\cup B$ be a generalized best proximity contraction mapping. Then the sequence $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are bounded, where $a_{n+1}=Fa_n, n=0,1,2,.....$ for any $a_0 \in A\cup B$

Proof: Let $a_0 \in A$. By theorem 2.3, $d(a_n,a_{n+1})$ converges to dist (A,B) .we prove that $\{a_{2n+1}\}$ is bounded. Suppose $\{a_{2n+1}\}$ is not bounded .Then there exits N_0 such that $d(F^2a_0, F^{2N_0+1}a_0) > K$ and $d(F^2a_0, F^{2N_0-1}a_0) \leq K$

$$K > \max\left\{\frac{1-(l_1+l_2+l_3)^2}{1-(l_1+l_2+l_3)^2}\, dist(A,B) + \frac{2(l_1+l_2+l_3)^2}{1-(l_1+l_2+l_3)^2} d(a_0,Fa_0),\ d(F^2a_0,Fa_0)\right\}$$

$K < d(F^2a_0, F^{2N_0+1}a_0)$ Using definition 2.1,we have

$$K \leq l_1 d(Fa_0, F^{2N_0}a_0) + l_2 d(Fa_0, B) + l_3 d(F^{2N_0}a_0, A) + (1-(l_1+l_2+l_3))\, dist(A,B)$$

$$= (l_1+l_2+l_3)\, d(a_1, a_{2N_0}) + (1-(l_1+l_2+l_3))\, dist(A,B)$$

$$K - dist(A,B) < (l_1+l_2+l_3)\,(d(a_1,a_{2N_0}) - dist(A,B))$$

$$\leq (l_1+l_2+l_3)\left[(l_1+l_2+l_3)\, d(a_0, F^{2N_0-1}a_0) + (1-(l_1+l_2+l_3))\, dist(A,B) - dist(A,B)\right]$$

$$= (l_1+l_2+l_3)^2\left[d(a_0,F^{2N_0-1}a_0) - dist(A,B)\right]$$

$$\frac{K - dist(A,B)}{(l_1+l_2+l_3)^2} + dist(A,B) < d(a_0, F^{2N_0-1}a_0)$$

$$\leq d(a_0, F^2 a_0) + d(F^2 a_0, F^{2N_0-1} a_0)$$

$$\leq \left[d(a_0, Fa_0) + d(Fa_0, F^2 a_0) \right] + d(F^2 a_0, F^{2N_0-1} a_0)$$

$$= \left[d(a_0, Fa_0) + d(a_0, Fa_0) \right] + d(F^2 a_0, F^{2N_0-1} a_0)$$

$$\leq 2d(a_0, Fa_0) + K$$

$$K\left(\frac{1}{(l_1+l_2+l_3)^2} - 1 \right) < dist(A,B) \left(\frac{1}{(l_1+l_2+l_3)^2} - 1 \right) + 2d(a_0, Fa_0)$$

$$K\left(\frac{1-(l_1+l_2+l_3)^2}{(l_1+l_2+l_3)^2} \right) < dist(A,B) \left(\frac{1-(l_1+l_2+l_3)^2}{(l_1+l_2+l_3)^2} \right) + 2d(a_0, Fa_0)$$

$$K\left(1-(l_1+l_2+l_3)^2\right) < dist(A,B)\left(1-(l_1+l_2+l_3)^2\right) + 2(l_1+l_2+l_3)^2 d(a_0, Fa_0)$$

$$K < \frac{1-(l_1+l_2+l_3)^2}{1-(l_1+l_2+l_3)^2} dist(A,B) + \frac{2(l_1+l_2+l_3)^2}{1-(l_1+l_2+l_3)^2} d(a_0, Fa_0)$$

Which is a contradiction. Hence $\{a_{2n+1}\}$ is bounded.

Theorem 2.5. Let A and B be non empty subsets of a uniformly convex Banach space X such that A is closed and convex . Let $F : A \cup B \to A \cup B$ be a generalized best proximity contraction mapping. Then $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are Cauchy sequences.

Proof: $d(Fa, Fb) = l_1 d(a,b) + l_2 d(a,\mathrm{B}) + l_3 d(\mathrm{A}, b) + (1-(l_1+l_2+l_3)) dist(A,B)$

We have

$$d(a_n, a_{n+1}) = (l_1+l_2+l_3) d(a_{n-1}, a_n) + (1-(l_1+l_2+l_3)) dist(A,B)$$

Case(i):

Assume $d(A,B) = 0$

$$d(a_n, a_{n+1}) \leq (l_1+l_2+l_3) d(a_{n-1}, a_n)$$

$$\leq (l_1+l_2+l_3)^2 d(a_{n-2}, a_{n-1})$$

..

..

$$\leq (l_1+l_2+l_3)^n d(a_0, a_1)$$

Hence, $\{a_{2n}\}$ is a Cauchy sequence and also $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are Cauchy sequences.

Case(ii): Suppose $d(A,B)>0$

it is enough to prove that a_{2n} is a Cauchy sequence.

Suppose not, there exits $\in_0 > 0,$ for each $l \geq 1$ there exits $m_l > n_l \geq l$ such that

$$d\left(a_{2m_l},a_{2n_l}\right) \geq \in_0 \tag{2.2}$$

Choose $0<\lambda<1$ such that $\frac{\in_0}{\lambda} > dist\left(A,B\right)$ and choose $\in > 0$ such that

$$\in < \min\left\{\frac{\in_0}{\lambda} - dist\left(A,B\right), \frac{dist\left(A,B\right)\delta\left(\lambda\right)}{1-\delta\left(\lambda\right)}\right\}$$

Claim: $d\left(a_{2m,}a_{2n+1}\right) < dist\left(A,B\right)+\in$

Suppose not, there exits $\in > 0$ such that for each $l \geq 1$ there exits $m_l \geq n_l \geq l$ such that

$$d\left(a_{2m_l,}a_{2n_l+1}\right) \geq dist\left(A,B\right)+\in$$

Now , $dist\left(A,B\right)+\in \leq d\left(a_{2m_l,}a_{2(n_l+1)}\right)$

$$\leq d\left(a_{2m_l},a_{2\left(m_l-1\right)}\right)+d\left(a_{2\left(m_l-1\right)},a_{2\left(n_l+1\right)}\right)$$

Since $d\left(a_{2m_l},a_{2\left(m_l-1\right)}\right)$ and $d\left(a_{2\left(m_l-1\right)},a_{2\left(n_k+1\right)}\right) \leq \in as l \to \infty$

Then $$\lim_{l\to\infty} d\left(a_{2m_l},a_{2(n_l+1)}\right) = d\left(A,B\right)+\in \tag{2.3}$$

Also $d\left(a_{2m_l,}a_{2n_l+1}\right) \leq \left(l_1+l_2+l_3\right)^2\left(dist\left(A,B\right)+\in\right)+\left(1-\left(l_1+l_2+l_3\right)^2\right)dist\left(A,B\right)$

$$= dist\left(A,B\right)+\left(l_1+l_2+l_3\right)^2 \in$$

Which is a contradiction. Hence $d\left(a_{2m,}a_{2n+1}\right) < dist\left(A,B\right)+\in$

Thus there exists N_1 such that

$$d\left(a_{2n_k},a_{2n_k+1}\right) < dist\left(A,B\right)+\in \tag{2.4}$$

for all $n_l \geq N$. Again there exists N_2 such that

$$d\left(a_{2m_l},a_{2n_l+1}\right) < dist\left(A,B\right)+\in \tag{2.5}$$

for all $m_l \geq N_1 \geq N_2$

Let $N = \max\left(N_1,N_2\right)$

Using 2.2 and 2.5 and uniform convexity of X, we obtain,

$$d\left(\frac{a_{2m_l}+a_{2n_l}}{2}, a_{2n_l+1}\right) \le \left(1-\delta\left(\frac{\in_0}{dist(A,B)+\in}\right)\right) dist(A,B)+\in \text{ for all } m_l > n_l \ge N,$$

By the choice of $\in$ and δ is increasing, we obtain,

$$d\left(\frac{a_{2m_l}+a_{2n_l}}{2}, a_{2n_l+1}\right) < dist(A,B) \text{ for all } m_l > n_l \ge N,$$

which is a contradiction .Hence $\{a_{2n}\}$ is a Cauchy sequence in A .Similarly, $\{a_{2n+1}\}$ is a Cauchy sequence in $B.$

Theorem 2.6. Let A and B be non empty subsets of a uniformly convex Banach space X such that A is convex . Let $F: A\cup B \to A\cup B$ be a generalized best proximity contraction mapping. For $a_0 \in A,$ define $a_{n+1} = Fa_n,$ for each $n \ge 0.$ Then $d(a_{2n+2}, a_{2n}) \to 0$ and $d(a_{2n+3}, a_{2n+1}) \to 0$ as $k \to \infty.$

Proof: To show that $d(a_{2n+2}, a_{2n}) \to 0$ as $k\to\infty.$ assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \in \mathbb{N}$, there exists $n_k \ge k$ so that

$$d(a_{2n_k+2}, a_{2n_k}) \ge \varepsilon_0 \tag{2.6}$$

Choose $0<\eta<1$ so that $\frac{\varepsilon_0}{\eta} > d(A,B)$ and choose ε such that

$$0<\varepsilon<\min\left\{\frac{\varepsilon_0}{\eta} - d(A,B), \frac{d(A,B)\delta'(\eta)}{1-\delta'(\eta)}\right\}.$$

By theorem 2.3, there exists N_1 such that

$$d(a_{2n_k+2}, a_{2n_k+1}) \le d(A,B)+\varepsilon \text{ for all } n_k \ge N_1. \tag{2.7}$$

Also, there exists N_2 such that

$$d(a_{2n_k+2}, a_{2n_k+1}) \le d(A,B)+\varepsilon \text{ for all } n_k \ge N_2. \tag{2.8}$$

Let $N = \max\{N_1, N_2\}$. It follows from (2.6),(2.8) and uniform convexity of X that

$$d\left(\frac{a_{2n_k+2}+a_{2n_k}}{2}, a_{2n_k+1}\right) \le \left(1-\delta'\left(\frac{\varepsilon_0}{d(A,B)+\varepsilon}\right)\right)(d(A,B)+\varepsilon),$$

for all $n_k \geq N.$ Since $\left(\frac{a_{2n_k+2}+a_{2n_k}}{2}\right) \in A,$ the choice of ε and the fact that $\delta^{'}$ is strictly increasing imply that $d\left(\frac{a_{2n_k+2}+a_{2n_k}}{2}, a_{2n_k+1}\right) < d\ \ A,B$, for all $a_k \geq N.$ which is a contradiction. A similar argument holds for $d\ \ a_{2n+3}, a_{2n+1}\ \ \to 0$ as $k \to \infty$.

Theorem 2.7. Let A and B be non empty closed subsets of a complete metric space X .Let $F: A\cup B \to A\cup B$ be a generalized best proximity contraction mapping, let $a_0 \geq A,$ and define $a_{n+1} = Fa_n.$ Suppose a_{2n} has a convergent subsequence in A .Then there exists a in A such that $d\ \ a, Fa\ \ = d\ \ A,B$.

Proof: Let a_{2n_k} be a subsequence of i a_{2n} converging to some $a \in A.$ Now,

$$d\ \ A,B\ \ \leq d\ \ a, a_{2n_k-1}\ \ \leq d\ \ a, a_{2n_k}\ \ + d\ \ a_{2n_k}, a_{2n_k-1}\ \ .$$

Taking limit as $k \to \infty$,

$$d\ \ A,B\ \ \leq \lim_{k\to\infty} d\ \ a, a_{2n_k-1}\ \ \leq d\ \ A,B\ \ .$$

Thus $d\ \ a, a_{2n_k-1}$ converges to $d\ \ A,B$ .Since

$$d\ \ A,B\ \ \leq \lim_{k\to\infty} d\ \ a_{2n_k}, Fa\ \ \leq d\ \ a_{2n_k-1}, a\ \ .$$

Again $k \to \infty$,

$$d\ \ A,B\ \ \leq \lim_{k\to\infty} d\ \ a_{2n_k}, Fa\ \ \leq d\ \ A,B\ \ .$$

Therefore, $d\ \ a, Fa\ \ = d\ \ A,B$.

Theorem 2.8. Let A and B be non empty closed and convex subsets of a uniformly convex Banach space. Suppose $F: A\cup B \to A\cup B$ be a generalized best proximity contraction mapping, then there exists a unique best proximity point a in A (that is with , $d\ \ a, Fa\ \ = d\ \ A,B$. For $a_0 \in A,$ define $a_{n+1} = Fa_n$ for each $n \geq 0$ then a_{2n} converges to the best proximity point.

Proof: Suppose $d\ \ A,B\ \ = 0,$ then $A \cap B \neq \phi$ and the theorem follows from Banach contraction theorem, as F is contraction map on $A \cap B$. Assume $d\ \ A,B\ \ \neq 0.$ Since $d\ \ a_{2n}, a_{2n+1}\ \ \to d\ \ A,B\ \ , d\ \ a_{2n+2}, a_{2n+1}\ \ \to d\ \ A,B$.

By theorem 2.6, $d(a_{2n}, a_{2n+1}) \to 0.$ Similarly we can show that $d(a_{2n+1}, a_{2n+3}) \to 0.$ Since, for each $\varepsilon > 0,$ there exists positive integer N_0 such that for all $m > n \geq N_0, d(a_{2m}, a_{2n+1}) < d(A, B) + \varepsilon.$ Therefore, $\{a_{2n}\}$ is a Cauchy sequence by theorem 2.5 and hence converges to same $a \in A.$ From theorem 2.7, it follows that $d(a, Fa) = d(A, B)$. To see that such a is unique, assume the contrary. Suppose $a, b \in A$ and $a \neq b$ such that $d(a, Fa) = d(A, B)$ and $d(b, Fb) = d(A, B)$.

Since, $d(F^2a, Fa) \leq d(Fa, a) = d(A, B)$, $F^2a = a$ and $F^2b = b.$

Therefore, $d(Fa, b) = d(Fa, F^2b) \leq d(a, Fb)$. $d(Fb, a) = d(Fb, F^2a) \leq d(b, Fa)$ which implies $d(Fb, a) = d(b, Fa)$. Since $a \neq b$ implies that $d(b, Fa) > d(A, B)$, it follows that

$$
\begin{aligned}
d(Fb, a) &= d(Fb, F^2a) \\
&\leq (l_1 + l_2 + l_3)\, d(b, Fa) + (1 - (l_1 + l_2 + l_3))\, d(A, B) \\
&< d(b, Fa)
\end{aligned}
$$

a contradiction. Therefore, $a = b.$ Hence the theorem.

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