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Certain Families of Generalized Mittag-Leffler Functions and their Integral Representation

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ABSTRACT

In this paper integral representation and some other results are established for some families of Mittag-Leffler function denoted by $E^{\gamma,q}_{\alpha,\beta}(z)$ and $E_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z]$ which are introduced and studied by Shukla and Prjapati and Saxena and Nishimoto respectively.

Keywords: Generalization of Mittag-Leffler function, integral representation and binomial expression.

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1. INTRODUCTION AND PRELIMINARIES

The entire function of the form

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{1.1}$$

Where α ∈ C ,Re (α) > 0 z ∈ C, defines the Mittag-Leffler function[5].
A Generalization of (1.1) in the form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.2}$$

Where α, β ∈ C; Re (α) > 0 , Re (β) > 0, z ∈ C is defined and studied by Wiman[3].

A Generalization of (1.2) is introduced in terms of series representation by Prabhakar[10] as

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\left(\frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)\mathrm{k}\,!}\right) \tag{1.3}$$

Where α, β, γ ∈ C; Re (α) > 0, Re (β) > 0 , Re (γ) > 0, z ∈ C and $(a)_n$ is the well-known Pochhammer symbol.
A Generalization of (1.3) is introduced by Shukla and Prajapati[2] in the following form

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\left(\frac{(\gamma)_{qn} z^n}{\mathrm{n}\,!\,\Gamma(\alpha n+\beta)}\right) \tag{1.4}$$

where α, β, γ ∈ C; Re (α) > 0 , Re (β) > 0, Re (γ) > 0, β > α >0 and q ∈ (0, 1) U N and $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol [4] which in particular reduces to $q^{qn}\prod_{r=1}^{q}[\frac{\gamma+r-1}{q}]_n$ if q∈ N.

Another Generalization of Mittag-Leffler function defined in (1.3) and (1.4) was introduced and studied by Srivastava and Tomovski[6] in the form

$$E_{\alpha,\beta}^{\gamma,K}(z) = \sum_{n=0}^{\infty}\left(\frac{(\gamma)_{nK} z^n}{n!\,\Gamma(\alpha n+\beta)}\right) \tag{1.5}$$

Where α, β, γ , K ∈ C, Re(α) > max{0,Re(K)-1}, Re (β) > 0 , Re (γ) > 0 Re (K) > 0 .

A further extension of both the Mittag-Leffler function defined in (1.5) and multi-index Mittag-Leffler function defined by Kriyakova[7,8] was recently introduced and studied by Saxena and Nishimoto[9] in the form

$$E_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z] = E_{\gamma,k}[(\alpha_1,\beta_1),\dots,\alpha_m,\beta_m));z]$$
$$= \sum_{r=0}^{\infty}\left(\frac{(\gamma)_{rk} z^r}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)\,r!}\right) \tag{1.6}$$

Where r, α_j, β_j, γ ∈ C; $Re(\alpha_j) > 0, Re(\beta_j) > 0, Re(k) > 0, (j = 1,2,\dots m)$.

When m=1 (1.6) reduces to (1.5).It is interesting to observe that for *γ=k=1,* (1.6) yields the multi-index Mittag-Leffler function, defined by Kriyakova[7] in the form

$$E_{1,1}\left[(\frac{1}{\alpha_j},\beta_j)_{1,m};z\right] = E_{\frac{1}{\alpha_j}}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\prod_{j=1}^{m}\Gamma\left(\beta_j+\frac{n}{\alpha_j}\right)(n)!}$$

Where α_j, β_j ∈ C; $Re(\alpha_j) > 0, Re(\beta_j) > 0, (j = 1,2,\dots m)$.

2. INTEGRAL REPRESENTATION OF $\mathbf{E_{\alpha,\beta}^{\gamma,q}(z)}$

In this section we obtained integral representation of the function $E_{\alpha,\beta}^{\gamma,K}(z)$ which is introduced by Shukla and Prajapati[2].

Theorem 2.1 If α, β, γ ∈ C; Re (α) > 0 , Re (β) > 0 , Re (γ) > 0, β > α >0 and q ∈ (0, 1) U N then

$$E_{\alpha,\beta}^{\gamma,q}(z) = \int_0^1 \frac{t^{\gamma+qn-1}(1-t)^{\alpha n+\beta-\gamma-qn-1}(1-z)^{-\gamma}}{\Gamma(\gamma+n)\Gamma(\alpha n+\beta-\gamma-qn)}dt$$

Proof. To prove the theorem 2.1we denote its right hand side by Δ_1 i.e.

$$\Delta_1 = \int_0^1 \frac{t^{\gamma+qn-1}(1-t)^{\alpha n+\beta-\gamma-qn-1}(1-z)^{-\gamma}}{\Gamma(\gamma+n)\Gamma(\alpha n+\beta-\gamma-qn)}dt$$

Now using binomial expression

$$\Delta_1 = \int_0^1\left[\frac{t^{\gamma+qn-1}(1-t)^{\alpha n+\beta-\gamma-qn-1}}{\Gamma(\gamma+n)\Gamma(\alpha n+\beta-\gamma-qn)}\sum_{n=0}^{\infty}\frac{z^n(\gamma)_n}{n!}\right]dt$$

Now interchanging the order of integration and summation we get

$$\Delta_1 = \sum_{n=0}^{\infty}\frac{z^n(\gamma)_n}{n!\,\Gamma(\gamma+n)\Gamma(\alpha n+\beta-\gamma-qn)}\int_0^1 t^{\gamma+qn-1}(1-t)^{\alpha n+\beta-\gamma-qn-1}dt$$

$$\Delta_1 = \sum_{n=0}^{\infty}\frac{z^n(\gamma)_n}{n!\,\Gamma(\gamma+n)\Gamma(\alpha n+\beta-\gamma-qn)}\frac{\Gamma(\gamma+qn)\Gamma(\alpha n+\beta-\gamma-qn)}{\Gamma(\alpha n+\beta)}$$

Using generalized Pochhammer symbol and solving we get

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma\,(\alpha n+\beta)\mathrm{n}\,!}$$
$$\Delta_1 = \mathrm{E}_{\alpha,\beta}^{\gamma,q}(\mathrm{z})$$

3. INTEGRAL REPRESENTATION OF $\mathbf{E}_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z]$

In this section we obtained integral representation of the function $\mathrm{E}_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z]$ which is introduced by Saxena and Nishimoto[9]

Theorem 3.1 If n, $\alpha_j, \beta_j, \gamma \in C; Re(\alpha_j) > 0, Re(\beta_j) > 0, Re(k) > 0, (j = 1,2, \ldots m)$ then

$$\mathrm{E}_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z] = \int_0^1 \frac{t^{\gamma+rk-1}(1-t)^{r-rk-1}(1-z)^{-\gamma}}{\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)\,\Gamma\,(\mathrm{r}-\mathrm{rk})}\,dt$$

Proof. To prove the theorem 3.1 we denote its right hand side by Δ_2 i.e.

$$\Delta_2 = \int_0^1 \frac{t^{\gamma+rk-1}(1-t)^{r-rk-1}(1-z)^{-\gamma}}{\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)\,\Gamma\,(\mathrm{r}-\mathrm{rk})}\,dt$$

Now using binomial expression

$$\Delta_2 = \int_0^1 \left[\frac{t^{\gamma+rk-1}(1-t)^{r-rk-1}}{\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)\,\Gamma\,(\mathrm{r}-\mathrm{rk})} \sum_{r=0}^{\infty}\frac{z^r(\gamma)_r}{r\,!}\right]dt$$

Now interchanging the order of integration and summation we have

$$\Delta_2 = \sum_{r=0}^{\infty} \frac{z^r(\gamma)_r}{\mathrm{r}\,!\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)\,\Gamma\,(r-rk)} \int_0^1 t^{\gamma+rk-1}(1-t)^{r-rk-1}\,dt$$

$$\Delta_2 = \sum_{r=0}^{\infty} \frac{z^r(\gamma)_r}{\mathrm{r}\,!\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)\,\Gamma\,(r-rk)} \frac{\Gamma\,(\gamma+rk)\Gamma\,(r-rk)}{\Gamma\,(\gamma+rk+r-rk)}$$

Using generalized Pochhammer symbol and solving we get

$$\Delta_2 = \sum_{r=0}^{\infty} \frac{(\gamma)_{rk} z^r}{\mathrm{r}\,!\prod_{j=1}^{m}\Gamma\,(\alpha_j r+\beta_j)}$$
$$\Delta_2 = \mathrm{E}_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z]$$

4. CERTAIN FAMILIES OF GENERALIZED MITTAG-LEFFLER FUNCTIONS

In this section we obtain some results related to generalized Mittag-Leffler functions denoted by $\mathrm{E}_{\alpha,\beta}^{\gamma,q}(\mathrm{z})$ and $\mathrm{E}_{\gamma,k}[(\alpha_j,\beta_j)_{1,m};z]$ which are introduced and studied by Shukla and

Prjapati[2] and Saxena and Nishimoto[9] respectively. To obtain the results we use binomial expression and gamma function. The results follows from the following theorems.

Theorem 4.1.If α, β, γ ∈ C; Re (α) > 0 , Re (β) > 0 , Re (γ) > 0, β > α >0 and q ∈ (0, 1) U N then

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{(\gamma)_{qn}(\lambda)_k z^{n+k}(1-z)^{\lambda}}{\mathrm{n}\,!\,\mathrm{k}\,!\,\Gamma\,(\alpha n+\beta)}\right) \tag{4.1}$$

Proof . To prove the result in 4.1 we denote its right hand side by Δ_3 i.e.

$$\Delta_3 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{(\gamma)_{qn}(\lambda)_k z^{n+k}(1-z)^{\lambda}}{\mathrm{n}\,!\,\mathrm{k}\,!\,\Gamma\,(\alpha n+\beta)}\right)$$

$$\Delta_3 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_k z^{k}(1-z)^{\lambda}}{\mathrm{k}\,!}\;\frac{(\gamma)_{qn} z^{n}}{\mathrm{n}\,!\,\Gamma\,(\alpha n+\beta)}$$

Now using binomial expression

$$\Delta_3 = \sum_{n=0}^{\infty}\left(\frac{(\gamma)_{qn} z^{n}}{\mathrm{n}\,!\,\Gamma\,(\alpha n+\beta)}\right)$$

$$\Delta_3 = E_{\alpha,\beta}^{\gamma,q}(z)$$

Theorem 4.2.If $r, \alpha_j, \beta_j \in C, Re(\alpha_j) > 0, Re(\beta_j) > 0, Re(k) > 0, (\,j = 1, \dots, m)$ then

$$\mathrm{E}_{\gamma,\mathrm{k}}[(\alpha_{\mathrm{j}}, \beta_{\mathrm{j}})_{1,\mathrm{m}}; \mathrm{z}] = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(\gamma)_{rk}(\lambda)_n z^{r+n}(1-z)^{\lambda}}{\mathrm{n}\,!\,\mathrm{r}\,!\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j\right)}\right) \tag{4.2}$$

Proof . To prove the result in 4.2 we denote its right hand side by Δ_4 i.e.

$$\Delta_4 = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(\gamma)_{rk}(\lambda)_n z^{r+n}(1-z)^{\lambda}}{\mathrm{n}\,!\,\mathrm{r}\,!\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j\right)}\right)$$

$$\Delta_4 = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_n z^{n}(1-z)^{\lambda}}{\mathrm{n}\,!}\;\frac{(\gamma)_{rk} z^{r}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j\right)\mathrm{r}\,!}$$

Now using binomial expression

$$\Delta_4 = \sum_{r=0}^{\infty} \left(\frac{(\gamma)_{rk} z^r}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j\right) \mathrm{r}\,!} \right)$$

$$\Delta_4 = \ \mathrm{E}_{\gamma,k}[(\alpha_j, \beta_j)_{1,m}; z]$$

Theorem 4.3 If α, β, γ ∈ C, a,b>0; Re (α) > 0, Re (β) > 0, Re (γ) > 0, β > α >0 and q ∈ (0, 1) U N then

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \int_0^{\infty} \left(\frac{m(ax)^m (\gamma)_{qn} z^n e^{-ax}}{\mathrm{n}\,!\ \Gamma\,(\mathrm{m}+1)\Gamma\,(\alpha\mathrm{n}+\beta) x} dx \right) \tag{4.3}$$

Proof . To prove the result in 4.3 we denote its right hand side by Δ_5 i.e.

$$\Delta_5 = \sum_{n=0}^{\infty} \int_0^{\infty} \left(\frac{m(ax)^m (\gamma)_{qn} z^n e^{-ax}}{\mathrm{n}\,!\ \Gamma\,(\mathrm{m}+1)\Gamma\,(\alpha\mathrm{n}+\beta) x} dx \right)$$

$$\Delta_5 = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n a^m}{\mathrm{n}\,!\,\Gamma\,(\alpha\mathrm{n}+\beta)\Gamma\,\mathrm{m}} \int_0^{\infty} x^{m-1}\, e^{-ax} dx$$

Now using definition of Gamma function we get

$$\Delta_5 = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n a^m}{\mathrm{n}\,!\,\Gamma\,(\alpha\mathrm{n}+\beta)\Gamma\,\mathrm{m}} \frac{\Gamma\,\mathrm{m}}{a^m}$$

Using (1.4) above equation immediately leads to $\Delta_5 = E_{\alpha,\beta}^{\gamma,q}(z)$

Theorem 4.4 If r, α_j, β_j, $\gamma \in C; Re(\alpha_j) > 0, Re(\beta_j) > 0, Re(k) > 0, (j = 1,2, \ldots m)$ then

$$E_{\gamma,k}[(\alpha_j, \beta_j)_{1,m}; z] = \sum_{r=0}^{\infty} \int_0^{\infty} \left(\frac{m(ax)^m (\gamma)_{rk} z^r e^{-ax}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j\right) \mathrm{r}!\, x\Gamma\,(\mathrm{m}+1)} dx \right) \tag{4.4}$$

Proof To prove the result in 4.4 we denote its right hand side by Δ_6 i.e

$$\Delta_6 = \sum_{r=0}^{\infty} \int_0^{\infty} \left(\frac{m(ax)^m (\gamma)_{rk} z^r e^{-ax}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j\right) \mathrm{r}!\, x\Gamma\,(\mathrm{m}+1)} dx \right)$$

$$\Delta_6 = \sum_{r=0}^{\infty} \frac{(\gamma)_{rk} z^r}{\prod_{j=1}^{m} \Gamma\,(\alpha_j r + \beta_j)\, \mathrm{r!}} \frac{a^m}{\Gamma\, \mathrm{m}} \int_0^{\infty} e^{-ax}\, x^{m-1}\, dx$$

Now using (1.6) and Gama function we get

$$\Delta_6 = E_{\gamma,k}[(\alpha_j, \beta_j)_{1,m}; z]$$

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