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Order of Convergence for the Solution of Fractional Integral Equation

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ABSTRACT

In this paper, we will estimate the order of the convergence for the solution of fractional integral equations such that the approximate solution converges to the exact solution. We will establish an estimation which imply the convergence of the method in function space for specific points.

Keywords: Fractional integral equation, Reproducing kernel method, Gram-Schmidt orthogonalization, Function space, Order of convergence.

1. INTRODUCTION

In this paper, our aim is to establish some error estimates of the reproducing kernel method in solving a class of Volterra fractional integral equation of the second kind which has not yet been discussed according to our knowledge. For this purpose, we consider the following non-linear Volterra fractional integral equation.

$u(x) = F(x, u(x))$

Where

$$F\big(x, u(x)\big) = f(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}\, k(x,\xi) G\big(u(\xi)\big) d\xi \qquad (1.1)$$

$0 < \xi \le x \le T_j < \infty, \qquad \alpha \ge o.$

in which functions f, k and the non-linear operator G are considered such that the equation (1.1) has a unique solution.

In the last decade, the reproducing kernel method has been applied for solving various problems such as ordinary differential equations, partial differential equations, difference equations and integral equations, which provides a new description forms for exact solutions and approximate solutions of many classical equations.

In recent years, some of the researchers develops the exact and approximate solution by reproducing kernel method such as, the expansion of exact solution and approximate solution for solving linear differential equation[5], an iterative algorithm for solving non-linear Volterra equations on the basis of the reproducing kernel Hilbert space without using the Gram-Schmidt orthogonalization process[2], numerical treatment of a class of fractional differential equations[1], reproducing kernel method for solving various boundary value problems[4], reproducing kernel with polynomial form is used for finding analytical and approximate solutions of a second - order hyperbolic equations[3].

We organize this paper as, in the second section some preliminaries are represented. In the third section estimation of the order of convergence for the solution of fractional Volterra integral equation are investigated.

2. TECHNICAL BACKGROUNDS

In this section, we use some definitions and theorems which are given in[2,4] with details and present technical preparation needed for the further discussion.

Definition 2.1 For a natural number m, the function space $W_m[0,T]$ is defined as follows $W_m[0,T] = \left\{\frac{u(x)}{u^{m-1}(x)}\right\}$ is absolutely continuous and $u^m(x) \in L^2[0,T]$.

The inner product and norm in $W_m[0,T]$ are defined respectively as follows

$$\langle u,v\rangle_{W_m} = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^T u^{(m)}(x)v^{(n)}(x)\,dx, \forall\, u,v \in W_m[0,T]$$

$\| u \|_{W_m} = \sqrt{\langle u,v\rangle_{W_m}}, \forall\, u, \in W_n[0,T]$

The function space $W_m[0,T]$ is actually a reproducing kernel space and its reproducing kernel $R_m(x,y)$has the property.

$u(x) = \langle u,(.),\rangle R_m(.,y)W_m, \forall\, u, \in W_n[0,T]$

The function space $W_1[0,1]$ and $W_2[0,1]$ are reproducing kernel spaces and their reproducing kernels are

$$R_1(x,y) = \left.\begin{matrix} 1+x, x \le y \\ 1+y, x > y \end{matrix}\right\}$$

And

$$R_2(x,y) = \left.\begin{matrix} 1 + yx + \frac{1}{2}yx^2 - \frac{1}{6}x^3, x \le y \\ 1 + xy + \frac{1}{2}xy^2 - \frac{1}{6}y^3, x > y \end{matrix}\right\}$$

For $m = 1, 2$ putting $\phi_i(x) = R_m(x, x_i)$ where $\{x_i\}_{i=1}^{\infty}$ is a set of points in the interval $[0, 1]$, the orthogonal system $\{\overline{\phi_i}(x)\}_{i=1}^{\infty}$can be derived from $\{\phi_i(x)\}_{i=1}^{\infty}$ by Gram-Schmidt ortogonalization process i. e.

$$\overline{\Phi_i}\,(x) = \sum_{k=1}^{i} \beta_{i\,k}\, \phi_k(x)$$

Theorem 2.1 Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[0, T]$. If equation (1.1) has a unique solution, then it can be represented in $W_m[0, T]$ as follows

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{i\,k} F(x_k\,, u(x_k))\overline{\Phi_i}\,(x) \tag{2.1}$$

Proof: Let

$u(x) = \sum_{j=1}^{\infty} A_j \overline{\Phi_j}\,(x)$,

where $A_j = \sum_{l=1}^{j} \beta_{jl}\, F(x_l, u(x_l\,))$

Now taking finitely many terms in the series representation of u as

$u_{mN}(x) = \sum_{j=1}^{N} \beta_j \overline{\Phi_j}\,(x)$

where $\beta_j = \sum_{l=1}^{j} \beta_{jl}\, F(x_l, u_{mj-1}(x_l\,))$

$\because u_{mN} \in W_m[0, T]$

We apply the method by following steps

(i)set $m = 1$ or 2

(ii)choose N collocation point in $[0, T]$

(iii)$\phi_i\,(x) = R_m(x, x_i), i = 1,2, \dots, N$

consider$\overline{\Phi_i}\,(x) = \sum_{k=1}^{i} \beta_{i\,k}\, \phi_k(x), i = 1,2, \dots, N$

(iv)initial function $u_{m0}(x)$

(v)set $n = 1$

(vi) set $\beta_n = \sum_{l=1}^{n} \beta_{nl}\, F(x_l, u(x_{m,n-1})(x_l)$

(vii) set $u_{mn}(x) = \sum_{j=1}^{n} \beta_j\, \overline{\Phi_j}\,(x)$

(viii) if $n < N$ then set $n = n + 1$ go to step (vi)

Theorem 2.2 If u_{1N} is bounded with respect to $W_1 -$ norm, then

$\| u - u_{1N} \|_{W_1} \to 0$, as $N \to \infty$.

Theorem 2.3 The approximate solution u_{1N} and its derivative u'_{2N} are both uniformly convergent.

3. MAIN RESULT

Estimation of the order of Convergence for the solution of Fractional Integral Equation:

For some positive integer N assume that $0 = x_1 < x_2 < \cdots \dots \dots \dots \dots \dots x_N = T$

And

$h_i = x_{i+1} - x_i \,, i = 1, 2, \dots \dots \dots \dots, N-1$

And

$h = \max_{1 \le i \le N-1} h_i$

Theorem 3.1 Let u and u_{1N} be the exact and approximate solution of (1.1) respectively. If $u \in C'[0,T]$ and $\| u'_{1N} \|_\infty \le M$, then

$\| u - u_{1N} \|_\infty \le Ch,$ C is constant

Proof: Assume $x \in [x_j, x_{j+1}]$ for $j = 1, 2, \dots \dots \dots N-1$

So

$$u(x) - u_{1N}(x) = u(x) - u(x_j) + u_{1N}\left(x_j - u_{1N}(x)\right) + u(x_j) - u_{1N}(x_j) \quad (3.1)$$

By the mean value theorem, we have

$u(x) - u(x_j) = (x - x_j)u'(\xi_j), \xi_j \in [x_j, x_{j+1}]$

$\because u \in C'[0,T]$, then for $M > 0$

$|u'(x)| \le M, \forall\, x \in [0,T]$

$|u(x) - u(x_j)| \le Mh,$

On the other hand, $u_{1N}(x) \in W_1[0,T]$ and we can write

$$u_{1N}(x_j) - u_{1N}(x) = -\int_{x_j}^{x} u'_{1N}(\xi)d\xi$$

Hence

$|u_{1N}(x_j) - u_{1N}(x)| \le M_1 h$

Using theorem (2.1), for large N, we have

$|u(x_j) - u_{1N}(x_j)| \le \epsilon \qquad (3.2)$

Now from equation (3.1) and (3.2)

$\| u - u_{1N} \|_\infty \le Ch$

Theorem 3.2 Let u and u_{2N} be the exact and approximate solutions of (1.1) respectively. If $u \in C^2[0,T]$ and $\| u''_{2N} \|_\infty \le M_2$, then

$\| u - u_{2N} \|_\infty \le Ch^2$, C is constant.

Proof: Assume $x \in [x_j, x_{j+1}]$ for $j = 1, 2, \dots \dots \dots N-1$

So

$$u'(x) - u'_{2N}(x) = u'(x) - u'(x_j) + u'_{2N}(x_j) - u'_{2N}(x) + u'(x_j) - u'_{2N}(x_j) \quad (3.3)$$

By the mean value theorem, we have

$u'(x) - u'(x_j) = (x - x_j)u''(\xi_j), \xi_j \in [x_j, x_{j+1}]$

$\because u \in C^2[0,T]$

Then

$|u''(x)| \le M_3, \forall\, x \in [0,T]$

Therefore

$\left|u'(x) - u'(x_j)\right| \leq M_3 h,$

On the other hand, $u_{2N} \in W_2[0, T]$

So

$$u'_{2N}(x_j) - u'_{2N}(x) = -\int_{x_j}^{x} u''_{2N}(\xi)d\xi$$

Hence

$|u'_{2N}(x_j) - u'_{2N}(x)| \leq M_2 h$

Using theorem (3.1), for large N, we have

$|u'(x_j) - u'_{2N}(x_j)| \leq \epsilon$

And

$$|u(x_j) - u_{2N}(x_j)| \leq \epsilon \qquad (3.4)$$

From equation (3.3) and (3.4)

$$\| u' - u'_{2N} \| \leq Ch, \text{where C is constant}$$

Using equation (3.4) and (3.5)

$$u(x) - u_{2N}(x) = u(x_j) - u_{2N}(x_j) + \int_{x_j}^{x} (u'(\xi) - u'_{2N}(\xi))\, d\xi$$

For the order of convergence of the method, using norm of infinity

$$E_{mN} = \| u - u_{mN} \|_{\infty} \simeq \max_{1 \leq i \leq N} | u(x_i) - u_{mN}(x_i)|, m = 1,2 \ldots ..$$

As

$$O_c = \frac{\ln\left(\frac{E_{mN}}{E_{mN/2}}\right)}{\ln(2)}$$

4. CONCLUSION

The reproducing kernel method in solving a class of the non-linear Volterra fractional integral equation of the second kind was investigated. Some error estimates was established and order of convergence $O(h)$ and $O(h^2)$ in the reproducing kernel space.

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