Journal of Computer and Mathematical Sciences, *Vol.8(1), 12-23 January 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Comparison of Higher Order Taylor's Method and Runge-Kutta Methods for Solving First Order Ordinary Differential Equations

Gashu Gadisa[1] and Habtamu Garoma[2]

[1,2]Department of Mathematics,
Jimma University, P.O. Box 378, Jimma, ETHIOPIA.
email: habte200@gmail.com

(Received on: January 12, 2017)

ABSTRACT

This paper mainly present, sixth order Taylor's method and fifth order Runge-Kutta method (RK5) for solving initial value problems of first order ordinary differential equations. The two proposed methods are quite efficient and practically well suited for solving these problems. In order to verify the accuracy, we compare numerical solutions with the exact solutions. The numerical solutions are in good agreement with the exact solutions. Numerical comparisons between Taylor's method and Runge-Kutta methods have been presented. The stability and convergence of the methods have been investigated. Two model examples (linear and non-linear) are given to demonstrate the reliability and efficiency of the methods. Point wise absolute errors are obtained by using MATLAB software. The proposed methods also compared with the existing literatures (RK4) and shows betterment results.

Keywords: Initial Value Problem, Taylor Series, Runge-Kutta Method, Stability Analysis.

1. INTRODUCTION

Many problems in science and engineering can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial-value problems (IVP) or boundary-value problems (BVP). Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using numerous methods, such as Taylor's method, Runge-Kutta method, and also some other methods. The Taylor's method is traditionally the first numerical technique. It is very simple to understand and geometrically easy to articulate but not very practical; the method has limited accuracy for more complicated functions. In the recent times, many authors have developed the Taylor's method for solving first and second order ordinary differential equations, for examples: Jorba and Zou[1], Miletics and Molnárka[2], Gibbons[3] and Moore[4]. The basic idea of these developments was the recursive calculation of the coefficients of the Taylor series.

A more robust and intricate numerical technique is the Runge-Kutta method. This method is the most widely used one since it gives reliable starting values and is particularly suitable when the computation of higher derivatives is complicated. The numerical results are very encouraging. Md. Islam[5] discussed accuracy analysis of numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE), Md. Islam[6] discussed accurate solutions of initial value problems for ordinary differential equations with fourth-order Runge-Kutta method. Ogunrinde *et al*.[7] studied on some numerical methods for solving initial value problems in ordinary differential equations. Gemechis File and Tesfaye Aga[8], considered the Runge-Kutta fourth order for solving quadratic Riccati differential equations. [9-14] also studied numerical solutions of initial value problems for ordinary differential equations using various numerical methods. In this paper, we introduce sixth order Taylor's method and fifth order RK method for solving first order ordinary differential equations.

2. PROBLEM FORMULATION

In this section we consider two numerical methods for finding the approximate solutions of the initial value problem (IVP) of the first-order ordinary differential equation has the form:

$$y' = f\left(x, y\left(x\right)\right), \quad x \in \left(x_0, x_n\right) \tag{1}$$

subject to the initial condition $y\left(x_0\right) = y_0$,

where $y' = \dfrac{dy}{dx}$ and $f\left(x, y\left(x\right)\right)$is a given function and $y(x)$ is the solution of the Eq. (1). In this paper we determine the solution of this equation on a finite interval$\left(x_0, x_n\right)$, starting with the initial point x_0. A continuous approximation to the solution $y(x)$, will not be obtained; instead, approximations to y will be generated at various values, called mesh points, in the interval$\left(x_0, x_n\right)$. Numerical methods employ the Eq. (1) to obtain approximations to the values of the solution corresponding to various selected values of $x_n = x_0 + nh,\ n = 1, 2, 3, \dots$. The parameter *h* is called the mesh size. The numerical solutions of Eq. (1) is given by a set of points $\left\{(x_n, y_n) : n = 0, 1, 2, ..., n\right\}$ and each point$\left(x_0, x_n\right)$, is an approximation to the corresponding point$\left(x_n, y(x_n)\right)$on the solution curve.

2.1. Taylor's Method

Let $y(x)$ is the exact solution of the Eq. (1). Now discretize the interval as: $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$. Then, $h = \frac{b-a}{n}$, where *n* is a positive integer.

The n^{th} Taylor Series expansion centered at x_i is given by:

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2!} f'(x_i, y(x_i)) + ... + \frac{h^n}{n!} f^{(n-1)}(x_i, y(x_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)), \quad \text{for } \xi_i \in [x_{i-1}, x_i] \tag{2}$$

$$\Rightarrow y_{i+1} = y_i + h\left(y'_i + \frac{h}{2!} y''_i + ... + \frac{h^{n-1}}{n!} y_i^{(n)} + \frac{h^n}{(n+1)!} y_i^{(n+1)}(\xi_i) \right)$$

$$\Rightarrow y_{i+1} = y_i + h\left(T_n(x_i, y(x_i)) + R_n(x_i)\right)$$

where,

$$T_n(x_i, y(x_i)) = y'_i + \frac{h}{2!} y''_i + ... + \frac{h^{n-1}}{n!} y_i^{(n)} = f(x_i, y(x_i)) + \frac{h}{2!} f'(x_i, y(x_i)) + ... + \frac{h^{n-1}}{n!} f^{(n-1)}(x_i, y(x_i))$$

$$R_n(x_i) = \frac{h^n}{(n+1)!} y_i^{(n+1)}(\xi_i)$$

Thus, the Taylor Method of order n (n^{th} order of Taylor Method) is:

$y_{i+1} = y_i + hT_n(x_i, y(x_i)), \text{ for } i = 1, 2, 3, ..., n\text{-}1$

The general formula for sixth order Taylor's method is given by:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!} y''_i + \frac{h^3}{3!} y'''_i + \frac{h^4}{4!} y_i^{(4)} + \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + O(h^7) \tag{3}$$

2.2. Runge-Kutta Method

This method was devised by two German mathematicians, Runge about 1894 and extended by Kutta a few years later. The Runge Kutta method is most popular because it is quite accurate, stable and easy to program. This method is distinguished by their order in the sense that they agree with Taylor's series solution up to terms of h^p where *p* is the order of the method. It do not demand prior computational of higher derivatives of $y(x)$ as in Taylor's series method. The fourth and fifth order Runge Kutta methods are commonly used for solving initial value problems (IVP) for ordinary differential equation (ODE). The fifth order Runge Kutta (RK5) for solving Eq. (1) is given by Nikolaos[13]:

$$y_{n+1} = y_n + \frac{1}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right) \tag{4}$$

where, y_{n+1} is the RK5 approximation of $y(x_n)$, and

$$k_1 = hf\ (x_n,\ y_n\ ),$$

$$k_2 = hf\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\ \left(x_n + \frac{h}{4},\ y_n + \frac{1}{16}\left(3k_1 + k_2\right)\right),$$

$$k_4 = hf\left(x_n + \frac{h}{2},\ y_n + \frac{k_3}{2}\right),$$

$$k_5 = hf\left(x_n + \frac{3h}{4},\ y_n + \frac{1}{16}\left(-3k_2 + 6k_3 + 8k_4\right)\right),$$

$$k_6 = hf\left(x_n + h,\ y_n + \frac{1}{7}\left(k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5\right)\right)$$

3. ERROR ANALYSIS FOR TAYLOR'S METHOD

Theorem: If Taylor method of order n is used to approximate the solution to the IVP $y' = f\left(x, y\left(x\right)\right)$, $x \in \left(x_0, x_n\right)$, $y\left(x_0\right) = y_0$ with mesh size h and $y \in C^{n+1}[x_0, y_0]$, then the local truncation error is $O(h^n)$.

Proof:
From Eq. (2) we have a local truncation error,

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(x_i, y(x_i)) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),\ \text{for}\ \xi_i \in \left[x_{i-1}, x_i\right].$$

So, if $y^{(n+1)}(x) = f^{(n)}(x, y(x))$ is bounded by $\left|y^{(n+1)}(x)\right| \le M$, for $x \in [a,b]$, then we get:

$$\left|\tau_{i+1}(h)\right| \le \frac{h^n}{(n+1)!} M = O(h^n)$$

Remark: The local truncation error in sixth order Taylor's method ($n = 6$) is evaluated as follows:

$$\left|\tau_{i+1}(h)\right| \le \frac{h^6}{5040} M\ ,\ \text{for all}\ i = 0,1,2,\ \ldots\ ,n$$

If $\left|y^{(6)}(x)\right| \le M$, for all $x \in [a,b]$, then $\left|\tau_{i+1}(h)\right| \le \dfrac{Mh^6}{5040} = O(h^6)$.

4. STABILITY ANALYSIS OF FIFTH ORDER RUNGE-KUTTA METHOD

Let the test differential equation of Eq. (1) is of the form:

$$y' = \lambda y, \quad y(x_0) = y_0 \tag{5}$$

where, λ is a constant, and has its solution in the form of,

$$y(x) = y(x_0)e^{\lambda(x-x_0)}$$

$$\Rightarrow y(x_n) = y(x_0)e^{\lambda nh} = y_0(e^{\lambda h})^n, \text{ at } x_n = x_0 + nh \tag{6}$$

Now, by considering the formula of RK5 in Eq. (4), we have:

$$k_1 = f(x_n, y_n) = \lambda h y_n$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = \left(\lambda h + \frac{(\lambda h)^2}{2}\right)y_n$$

$$k_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{3k_1 + k_2}{16}\right) = \left(\lambda h + \frac{(\lambda h)^2}{64} + \frac{(\lambda h)^3}{32}\right)y_n$$

$$k_4 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_3}{2}\right) = \left(\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{128} + \frac{(\lambda h)^4}{64}\right)y_n$$

$$k_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{1}{16}\left(-3k_2 + 63k_3 + 8k_4\right)\right)$$

$$= \left\{\lambda h + \frac{3}{4}(\lambda h)^2 + \frac{99}{512}(\lambda h)^3 + \frac{33}{2048}(\lambda h)^4 + \frac{9}{1024}(\lambda h)^5\right\}y_n$$

$$k_6 = hf\left(x_n + h, y_n + \frac{1}{7}\left(k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5\right)\right)$$

$$= \left\{\lambda h + (\lambda h)^2 + \frac{67}{224}(\lambda h)^3 + \frac{105}{448}(\lambda h)^4 - \frac{15}{1792}(\lambda h)^5 + \frac{9}{896}(\lambda h)^6\right\}y_n$$

On substituting the values of $k_1, k_2, ..., k_6$, in RK5 of Eq. (4), we obtain:

$$y_{n+1} = \left\{1 + \frac{7}{90}\lambda h y_n + \frac{16}{45}\left(\lambda h + \frac{(\lambda h)^2}{64} + \frac{(\lambda h)^3}{32}\right) + \frac{2}{15}\left(\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{128} + \frac{(\lambda h)^4}{64}\right)\right.$$
$$+ \frac{16}{45}\left(\lambda h + \frac{3}{4}(\lambda h)^2 + \frac{99}{512}(\lambda h)^3 + \frac{33}{2048}(\lambda h)^4 + \frac{9}{1024}(\lambda h)^5\right)$$
$$\left. + \frac{7}{90}\left(\lambda h + (\lambda h)^2 + \frac{67}{224}(\lambda h)^3 + \frac{105}{448}(\lambda h)^4 - \frac{15}{1792}(\lambda h)^5 + \frac{9}{896}(\lambda h)^6\right)\right\}y_n$$

$$\Rightarrow y_{n+1} = E(\lambda h)y_n$$

where,

$$\begin{aligned}E(\lambda h)=\Bigg\{1+\frac{7}{90}\lambda h+\frac{16}{45}\left(\lambda h+\frac{(\lambda h)^2}{64}+\frac{(\lambda h)^3}{32}\right)+\frac{2}{15}\left(\lambda h+\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^3}{128}+\frac{(\lambda h)^4}{64}\right)\\ +\frac{16}{45}\left(\lambda h+\frac{3}{4}(\lambda h)^2+\frac{99}{512}(\lambda h)^3+\frac{33}{2048}(\lambda h)^4)+\frac{9}{1024}(\lambda h)^5\right)\\ +\frac{7}{90}\left(\lambda h+(\lambda h)^2+\frac{67}{224}(\lambda h)^3+\frac{105}{448}(\lambda h)^4-\frac{15}{1792}(\lambda h)^5+\frac{9}{896}(\lambda h)^6\right)\Bigg\}\end{aligned} \tag{7}$$

From Eq. (6), it is easily observed that the exact value of $y(x_n)$ increases for the constant $\lambda>0$ and decreases for $\lambda<0$ with the factor $e^{\lambda h}$. While, from Eq, (7) the approximate value of y_n increases or decreases with the factor of $E(e^{\lambda h})$.

If $\lambda h>0$, then $e^{\lambda h}>1$, so the fifth order Runge-Kutta method in Eq. (4) is relatively stable. If $\lambda h<0$ $(i.e,\ \lambda<0,\text{since}\ h>0)$, then the interval of absolutely stable is $-2.99<\lambda h<0.$

5. NUMERICAL EXAMPLES

To validate the applicability of the methods, two examples with initial conditions have been considered. For each number of nodal points N, the point wise absolute errors are approximated by the formula, $\|E\|=|y(x_i)-y_i|$, for $i=0,1,2,\cdots,N$, where $y(x_i)$ and y_i are the exact and computed approximate solution of the given problems respectively, at the nodal point x_i. Numerical examples are given to illustrate the efficiency and convergence of the methods.

Example 1: Consider the initial value problem $y'(x)=x^2+xy,\ y(0)=1$ on the interval $0\le x\le 1.$ The exact solution of the given problem is given by

$$y(x)=\sqrt{\frac{\pi}{2}}e^{x^2/2}erf\left(\frac{x}{\sqrt{2}}\right)+e^{x^2/2}-x.$$

The approximate results and point wise absolute errors are obtained and shown in Tables (1-4) for different values of mesh size.

Example 2: Consider the initial value problem $y'(x)=xy-y^2,\ y(0)=1$ on the interval $0\le x\le 1.$ The exact solution of the given problem is given by $y(x)=\dfrac{2e^{x^2/2}}{\sqrt{2\pi}erfi\left(\frac{x}{\sqrt{2}}\right)+2}$.

The approximate results and point wise absolute errors are obtained and shown in Tables (5-8) for different values of mesh size.

6. NUMERICAL RESULTS

Table 1: Numerical approximations and absolute errors for Example 1 with step size $h = 0.1$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	1.005346521812841	4.16045e−08	1.005346520833333	9.7951e-10	1.005346522266007	4.5317e-10
0.2	1.022889462475293	8.26716e−08	1.022889460240387	2.2349e-09	1.022889463386417	9.1112e-10
0.3	1.055191963766034	1.23029e−07	1.055191959869256	3.8968e-09	1.055191965152622	1.3866e-09
0.4	1.105318952970661	1.63325e−07	1.105318946826345	6.1443e-09	1.105318954860455	1.8898e-09
0.5	1.176974972518977	2.05805e−07	1.176974963290857	9.2281e-09	1.176974974941074	2.4221e-09
0.6	1.274678991977672	2.55624e−07	1.274678978474642	1.3503e-08	1.274678994941241	2.9636e-09
0.7	1.403988318400775	3.23030e−07	1.403988298924772	1.9476e-08	1.403988321850967	3.4502e-09
0.8	1.571787769675660	4.26941e−07	1.571787741799152	2.7877e-08	1.571787773408868	3.7332e-09
0.9	1.786665853619037	6.00769e−07	1.786665813858258	3.9761e-08	1.786665857127219	3.5082e-09
1.0	2.059407405342576	9.01815e−07	2.059407348675452	5.6667e-08	2.059407407535726	2.1931e-09

Table 2: Numerical approximations and absolute errors for Example 1 with step size $h = 0.05$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	1.005346521812841	2.59745e−09	1.005346521796813	7.5442e-12	1.005346521828092	1.5251e-11
0.2	1.022889462475293	5.15419e−09	1.022889462438579	1.6028e-11	1.022889462506138	3.0845e-11
0.3	1.055191963766034	7.64925e−09	1.055191963701786	3.6714e-11	1.055191963813315	4.7282e-11
0.4	1.105318952970660	1.01097e−08	1.105318952869012	6.4247e-11	1.105318953035686	6.5026e-11
0.5	1.176974972518977	1.26597e−08	1.176974972365814	1.0165e-10	1.176974972603291	8.4314e-11
0.6	1.274678991977672	1.56044e−08	1.274678991752849	1.5316e-10	1.274678992082454	1.0478e-10
0.7	1.403988318400775	1.95705e−08	1.403988318075498	2.2482e-10	1.403988318525527	1.2475e-10
0.8	1.571787769675660	2.57387e−08	1.571787769208655	3.2528e-10	1.571787769815624	1.3996e-10
0.9	1.786665853619038	3.62246e−08	1.786665852950904	4.6700e-10	1.786665853760351	1.4131e-10
1.0	2.059407405342576	5.46971e−08	2.059407404387435	6.6813e-10	2.059407405453518	1.1094e-10

Table 3: Numerical approximations and absolute errors for Example 1 with step size $h = 0.025$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	1.005346521812841	1.62280e−10	1.005346521812585	2.5624e-13	1.005346521813335	4.9405e-13
0.2	1.022889462475293	3.21760e−10	1.022889462474705	5.8753e-13	1.022889462476295	1.0016e-12
0.3	1.055191963766034	4.76790e−10	1.055191963765003	1.0305e-12	1.055191963767574	1.5408e-12
0.4	1.105318952970660	6.28600e−10	1.105318952969027	1.6334e-12	1.105318952972788	2.1276e-12
0.5	1.176974972518977	7.84350e−10	1.176974972516512	2.4649e-12	1.176974972521750	2.7736e-12
0.6	1.274678991977673	9.62520e−10	1.274678991974048	3.6247e-12	1.274678991981143	3.4708e-12
0.7	1.403988318400775	1.20166e−09	1.403988318395523	5.2522e-12	1.403988318404949	4.1738e-12
0.8	1.571787769675661	1.57534e−09	1.571787769668108	7.5533e-12	1.571787769680416	4.7546e-12
0.9	1.786665853619039	2.21636e−09	1.786665853608216	1.0823e-11	1.786665853623978	4.9383e-12
1.0	2.059407405342576	3.35651e−09	2.059407405327083	1.5493e-11	2.059407405346751	4.1744e-12

Table 4: Numerical approximations and absolute errors for Example 1 with step size $h = 0.0125$

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	1.005346521812841	1.01399e−11	1.005346521812837	3.7748e-15	1.005346521812857	1.6209e-14
0.2	1.022889462475293	2.00999e−11	1.022889462475283	9.5479e-15	1.022889462475325	3.2196e-14
0.3	1.055191963766034	2.97600e−11	1.055191963766018	1.5987e-14	1.055191963766083	4.9294e-14
0.4	1.105318952970660	3.91900e−11	1.105318952970634	2.5979e-14	1.105318952970728	6.7724e-14
0.5	1.176974972518977	4.88001e−11	1.176974972518938	3.9302e-14	1.176974972519066	8.8374e-14
0.6	1.274678991977672	5.97400e−11	1.274678991977615	5.7510e-14	1.274678991977784	1.1147e-13
0.7	1.403988318400774	7.44000e−11	1.403988318400692	8.1934e-14	1.403988318400909	1.3523e-13
0.8	1.571787769675658	9.73599e−11	1.571787769675541	1.1768e-13	1.571787769675814	1.5588e-13
0.9	1.786665853619035	1.36930e−10	1.786665853618866	1.6920e-13	1.786665853619199	1.6431e-13
1.0	2.059407405342576	2.07670e−10	2.059407405342330	2.4647e-13	2.059407405342716	1.3944e-13

Table 5: Numerical approximations and absolute errors for Example 2 with step size $h = 0.1$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	0.913509127898782	1.95878e−07	0.913509390277778	2.6238e-07	0.913509122308024	5.5908e-09
0.2	0.849218518702443	3.47642e−07	0.849218853602540	3.3490e-07	0.849218512870538	5.8319e-09
0.3	0.801823397957602	4.53096e−07	0.801823740655947	3.4270e-07	0.801823393191519	4.7661e-09
0.4	0.767783586159507	5.24033e−07	0.767783915049449	3.2889e-07	0.767783582576678	3.5828e-09
0.5	0.744689700478634	5.72232e−07	0.744690009625120	3.0915e-07	0.744689697882640	2.5960e-09
0.6	0.730888402778509	6.06628e−07	0.730888692006850	2.8923e-07	0.730888400925731	1.8528e-09
0.7	0.725251299272098	6.33400e−07	0.725251570396258	2.7112e-07	0.725251297944772	1.3273e-09
0.8	0.727027086217658	6.56635e−07	0.727027341564614	2.5535e-07	0.727027085236218	9.8144e-10
0.9	0.735743588544958	6.78965e−07	0.735743830378873	2.4183e-07	0.735743587763266	7.8169e-10
1.0	0.751140351957987	7.02025e−07	0.751140582266148	2.3031e-07	0.751140351255600	7.0239e-10

Table 6: Numerical approximations and absolute errors for Example 2 with step size $h = 0.05$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	0.913509127898782	6.58113e−09	0.913509131263380	3.3646e-09	0.913509127898782	1.8735e-10
0.2	0.849218518702443	1.38536e−08	0.849218523026315	4.3239e-09	0.849218518702443	2.0138e-10
0.3	0.801823397957602	1.97359e−08	0.801823402402744	4.4451e-09	0.801823397957602	1.7055e-10
0.4	0.767783586159507	2.41269e−08	0.767783590439157	4.2796e-09	0.767783586159507	1.3389e-10
0.5	0.744689700478634	2.73825e−08	0.744689704510264	4.0316e-09	0.744689700478634	1.0234e-10
0.6	0.730888402778509	2.98840e−08	0.730888406556233	3.7777e-09	0.730888402778509	7.7985e-11
0.7	0.725251299272098	3.19353e−08	0.725251302817287	3.5452e-09	0.725251299272098	6.0329e-11
0.8	0.727027086217658	3.37550e−08	0.727027089559267	3.3416e-09	0.727027086217658	4.8321e-11
0.9	0.735743588544958	3.54931e−08	0.735743591711681	3.1667e-09	0.735743588544958	4.0978e-11
1.0	0.751140351957987	3.72487e−08	0.751140354975232	3.0172e-09	0.751140351957987	3.7512e-11

Table 7: Numerical approximations and absolute errors for Example 2 with step size $h = 0.025$.

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	0.913509127898782	2.59029e−10	0.913509127946245	4.7462e-11	0.000000000006011	6.0106e-12
0.2	0.849218518702443	6.51469e−10	0.849218518763660	6.1217e-11	0.000000000006540	6.5398e-12
0.3	0.801823397957602	9.97784e−10	0.801823398020694	6.3092e-11	0.000000000005615	5.6146e-12
0.4	0.767783586159507	1.26955e−09	0.767783586220354	6.0847e-11	0.000000000004477	4.4774e-12
0.5	0.744689700478634	1.47866e−09	0.744689700536023	5.7390e-11	0.000000000003484	3.4842e-12
0.6	0.730888402778509	1.64401e−09	0.730888402832330	5.3821e-11	0.000000000002710	2.7096e-12
0.7	0.725251299272098	1.78223e−09	0.725251299322637	5.0539e-11	0.000000000002142	2.1415e-12
0.8	0.727027086217658	1.90588e−09	0.727027086265316	4.7658e-11	0.000000000001750	1.7504e-12
0.9	0.735743588544958	2.02387e−09	0.735743588590137	4.5179e-11	0.000000000001506	1.5060e-12
1.0	0.751140351957987	2.14228e−09	0.751140352001045	4.3058e-11	0.000000000001383	1.3834e-12

Table 8: Numerical approximations and absolute errors for Example 2 with step size $h = 0.0125$

x_i	Exact Solution	Md.Amirul[14] (RK4)	6th Order Taylor's Method		5th Order Runge-Kutta	
			Numerical solution (y_i)	Absolute Error	Numerical solution (y_i)	Absolute Error
0.1	0.913509127898782	1.17910e−11	0.913509127899487	7.0455e-13	0.913509127898782	1.8974e-13
0.2	0.849218518702443	3.44851e−11	0.849218518703353	9.0972e-13	0.849218518702443	2.0772e-13
0.3	0.801823397957602	5.54771e−11	0.801823397958542	9.3936e-13	0.801823397957602	1.7941e-13
0.4	0.767783586159507	7.23600e−11	0.767783586160414	9.0683e-13	0.767783586159507	1.4422e-13
0.5	0.744689700478634	8.55770e−11	0.744689700479489	8.5565e-13	0.744689700478634	1.1302e-13
0.6	0.730888402778509	9.61640e−11	0.730888402779311	8.0269e-13	0.730888402778509	8.8485e-14
0.7	0.725251299272098	1.05089e−10	0.725251299272853	7.5451e-13	0.725251299272098	7.0277e-14
0.8	0.727027086217657	1.13103e−10	0.727027086218369	7.1199e-13	0.727027086217657	5.7510e-14
0.9	0.735743588544957	1.20751e−10	0.735743588545633	6.7546e-13	0.735743588544957	4.9516e-14
1.0	0.751140351957987	1.28405e−10	0.751140351958630	6.4293e-13	0.751140351957987	4.6296e-14

The following Figs. (1-4) shows the numerical solutions obtained by sixth order Taylor's method (TM6) and fifth order Runge-Kutta (RK5) versus the corresponding exact solution and Figs. (5-6) shows the numerical solutions obtained by the present methods with different mesh size h.

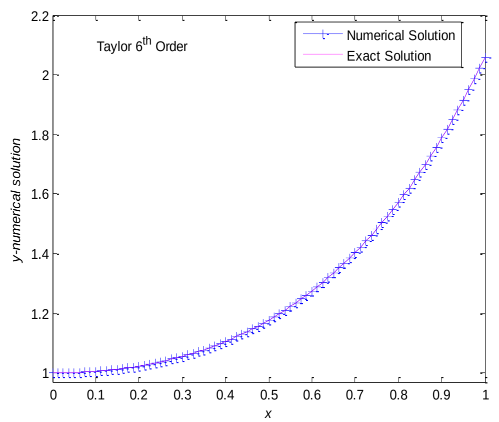

Fig 1: Numerical solution of Example 1, TM6 for $N = 80$

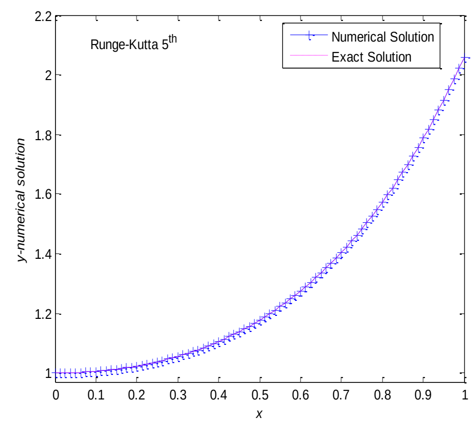

Fig 2: Numerical solution of Example 1, RK5 for $N = 80$

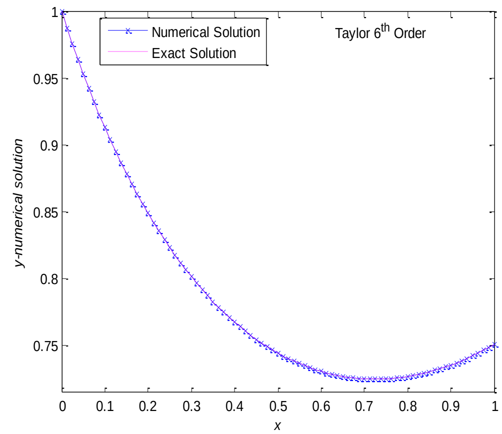

Fig 3: Numerical solution of Example 2, TM6 for $N = 80$

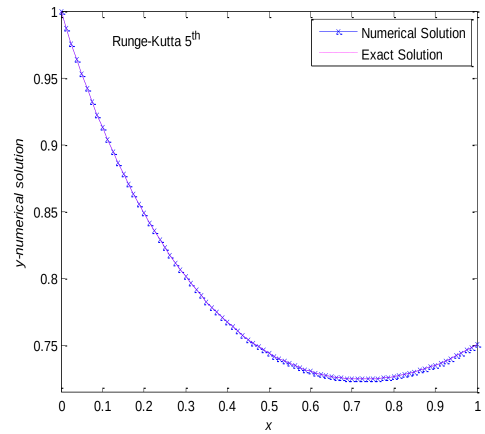

Fig 4: Numerical solution of Example 2, RK5 for $N = 80$

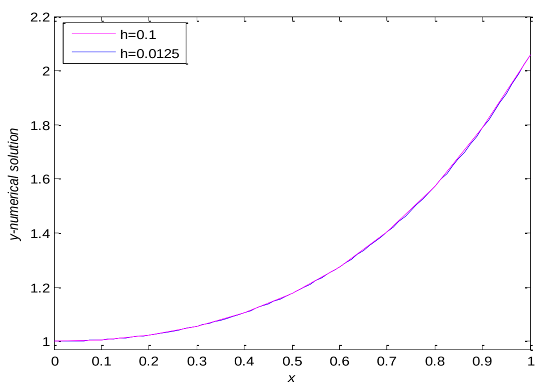

Fig 5: Numerical solution of Example 1 by TM6 for different values of mesh size *h*.

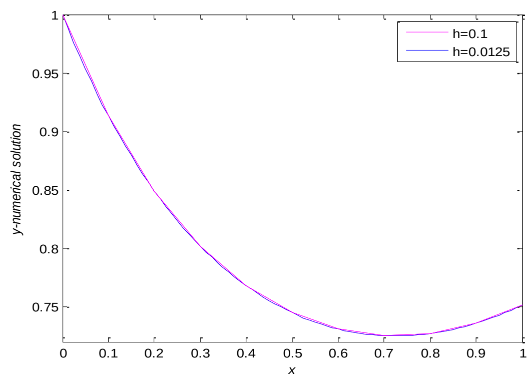

Fig 6: Numerical solution of Example 2 by RK5 for different values of mesh size *h*.

The Rate of Convergence for the Present Methods

The computational rate of convergence can also be obtained by using the double mesh principle defined below. Consider the numerical solution obtained by Eqs. (3) and (4) and let $Z_h = \max_i \left| y_i^h - y_i^{h/2} \right|$, $i = 1, 2, ..., N-1$, where y_i^h is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at the nodal point x_i and $x_i = x_0 + ih$, $i = 1, 2, ..., N-1$ and $y_i^{h/2}$ is the numerical solution at the nodal point x_i on the mesh $\{x_i\}_1^{2N-1}$ where, $x_i = x_0 + i\,h/2$, $i = 1, 2, ..., 2N-1$ (*i.e.*, the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals).

In the same way one can define $Z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$, that is:

$$Z_{h/2} = \max_i \left| y_i^{h/2} - y_i^{h/4} \right|, \text{ for } i = 1, 2, ..., 2N-1.$$

The computational rate of convergence ρ is also obtained by using the double mesh principle defined as, Doolan *et al.*[15]:

$$\rho = \frac{\left(\log\left(Z_h\right) - \log\left(Z_{h/2}\right)\right)}{\log 2}.$$

In Tables 9 and 10, the rate of convergence for Examples 1 and 2 respectively for both sixth order Taylor's method and fifth order RK method are given at different mesh sizes.

Table 9: Rate of convergence for model examples at $x = 0.3$ with different mesh sizes

	N	10	20	30	40	50
Rate of convergence for TM6	Example 1	5.9225	5.9622	5.9787	6.0103	6.1676
	Example 2	6.2686	6.1386	6.0932	6.0696	6.0547

Table 10: Rate of convergence for model examples at $x = 0.3$ with different mesh sizes

	N	10	20	30	40	50
Rate of convergence for RK5	Example 1	4.8741	4.9396	4.9576	4.9661	4.9529
	Example 2	4.8046	4.9248	4.9543	4.9678	4.9745

7. DISCUSSION AND CONCLUSION

Comparison of higher order Taylor's method and Runge-Kutta methods for solving first order ordinary differential equation with initial condition have been presented. Two model examples of different kinds of ordinary differential equations have been considered to verify the proposed formulae for different values of the mesh size *h*. The numerical solutions are tabulated (Tables 1-8) in terms of point wise absolute errors and one can observed that the present methods (i.e., sixth order Taylor's method and fifth order Runge-Kutta method) improves the findings of Md. Amirul[14] (RK4). Furthermore, RK5 is more accurate than RK4 and for non-linear first order ordinary differential equation (Example 2), RK5 is more accurate than the sixth order Taylor's method (Tables 5-8) and it is the most powerful and effective method for solving initial value problems for ordinary differential equations.

Furthermore, it is significant that all of the absolute errors decrease rapidly as *h* decreases for both methods. This shows that the small mesh size provides the better approximation. The stability regions of RK5 and error analysis of TM6 have been investigated. The results presented in Tables (9-10) confirmed that computational rate of convergence as well as a theoretical estimate indicates, sixth order Taylor's method is a sixth order convergent and RK5 is fifth order convergent. Figs. (1-4) shows that both methods approximate the exact solution very well and Figs. (5-6) shows that both methods are applicable for different values of mesh size *h.*

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