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Characterization of Zero-inflated Gamma Distribution

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ABSTRACT

Distributions are useful to model random phenomena. New random experiments are conducted and new data sets are encountered. In turn, new distributions emerge. Zero-inflated discrete models are such examples and they are useful in various situations. Further, they have also been characterized [see Nanjundan and Sadiq Pasha (2018)]. Zero-inflated continuous distributions are discussed in the context of insurance portfolio. Zero-inflated gamma distribution is characterized in this paper through a differential equation satisfied by its moment generating function.

Keywords: Zero-inflated discrete distributions, zero-inflated gamma distribution, moment generating function, linear differential equation.

1. INTRODUCTION

Characterization of a distribution is studying a unique property enjoyed by it. A probability distribution can be characterized through various methods. Lack of memory property characterizes exponential distribution in the continuous case and geometric distribution in the discrete case. Distributions have been characterized through the lower bounds of certain functions of their variances [see Srivastava and Sreehari (1990)]. Also, life time distributions are characterized by mean residual life function [see Sankaran and Unnikrishnan (1993)]. Further, the characterization of continuous distributions by truncated moments has been addressed by Ahsanullah *et al*. (2016).

It is highly infeasible in a paper like this to summarize all types of characterizations of distributions. Since zero-inflated gamma distribution is characterized through a differential equation, a brief review of literature in this direction is presented.

Nanjundan (2011) has characterized a subfamily of power series distributions through a differential equation satisfied by the probability generating functions (pgfs) of the distributions. Nanjundan and Sadiq Pasha (2015a, 2015b) have characterized zero-inflated Poisson and zero-inflated binomial distributions through a linear differential equation. Suresh *et al*. (2015) have identified a linear differential equation that characterizes zero-inflated negative binomial distribution. Along the same lines, Nanjundan and Sadiq Pasha (2018) have characterized a subfamily of zero-inflated power series distributions via a differential equation satisfied by their pgfs.

If a general insurance policy is under the detectable agreement, a claim will not be recorded and honored unless the loss exceeds a prescribed detectable limit. Also, if an insured does not claim for any loss, there will be a bonus for the next year's premium. Naturally there is a huge hunger for such a bonus. Hence the policy holders do not report all their losses in order to gain bonus for the next year. Therefore, a data set on the claim counts of an insurance portfolio will have excess zeros. Yip and Yau (2005) have used zero-inflated Poisson and zero-inflated negative binomial distributions to model claim count data, Boucher *et al*. (2009) have employed zero-inflated distributions to model insurance panel count data. They have also illustrated zero-inflated models based on the data on the claims reported to a Spanish insurance company.

In the context of both life and nonlife insurance, the claim sizes are usually modeled by continuous distributions with the positive support. But due to the hunger for bonus on no claim, a data set on the claim sizes of an insurance portfolio will have excess zeros. This leads to distributions which are mixtures of a distribution degenerate at zero and another one that is continuous with positive support. Such distributions are called perturbed continuous distributions or zero-inflated continuous distributions [see Nanjundan (2005), Mills (2013)]. In this paper, we are interested in zero-inflated gamma distribution.

2. ZERO-INFLATED GAMMA DISTRIBUTION

Let X be a random variable with $P(X=0)=\varphi$, $0<\varphi<1$ and having a probability density function (pdf) $(1-\varphi)f(x)$ for $x>0$, where $f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta x}x^{\alpha-1}, \alpha, \beta>0$. Then, the distribution function of X is given by

$$F(x)=P(X\leq x)=\begin{cases} 0, & x<0 \\ \varphi, & x=0 \\ (1-\varphi)\int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}dy, & x>0. \end{cases} \tag{2.1}$$

Thus the distribution of X is a zero-inflated gamma distribution with parameters φ, α, and β. The moment generating function (mgf) of X is given by

$$M(t)=\int_0^\infty e^{tx}dF(x), -\infty<t<\infty$$

$$=\varphi+(1-\varphi)\int_0^\infty e^{tx}f(x)dx$$

$= \varphi + (1-\varphi)\frac{\beta^{\alpha}}{\gamma(\alpha)}\int_0^{\infty} e^{-(\beta-t)x}x^{\alpha-1}dx$

$= \varphi + (1-\varphi)\frac{\beta^{\alpha}}{\gamma(\alpha)}\frac{\gamma(\alpha)}{(\beta-t)^{\alpha}}$

$M(t) = \varphi + \frac{(1-\varphi)}{(1-\frac{t}{\beta})^{\alpha}}.$ (2.2)

As a special case when $\alpha = 1$, zero-inflated gamma distribution reduces to zero-inflated exponential distribution and the corresponding mgf is given by

$M(t) = \varphi + \frac{(1-\varphi)}{(1-\frac{t}{\beta})}.$

3. CHARACTERIZATION

The following theorem characterizes zero-inflated gamma distribution.

Theorem: Let X be a nonnegative random variable with the pdf $f(x)$ for $x > 0$. If the moment generating function of X is such that

$M(t) = a + \frac{1}{b}(c-t)M'(t),$ (3.1)

where $0 < a < 1$, and $b, c > 0$ are constants, $t < c$, and $M'(t)$ is the derivative of $M(t)$, then X is zero-inflated gamma random variable.

Proof: It is straight forward to verify that the mgf of zero-inflated gamma distribution satisfy (3.1).

1) Suppose that X has zero-inflated gamma distribution with the distribution function (df) specified in (2.1). Then the mgf of X is given by

$M(t) = \varphi + \frac{(1-\varphi)}{(1-\frac{t}{\beta})^{\alpha}}.$

On differentiating the mgf, we get

$M'(t) = \frac{(1-\varphi)\alpha}{\beta(1-\frac{t}{\beta})^{\alpha+1}}$

and

$M(t) = \varphi + \frac{1}{\alpha}(\beta - t)M'(t).$ (3.2)

Hence $M(t)$ in (3.2) satisfies (3.1) with $a = \varphi$, $b = \alpha$, and $c = \beta$.

2) Suppose that the mgf of X satisfies the linear differential equation in (3.1). Now let us have a close look at the possible values of b, c and their consequences.

i) If $b = 0$, then (3.1) turns out be $M(t) = \infty$, $\forall\ t \in R$ and hence $M(t)$ has no meaning. Therefore $b \neq 0$.

ii) Let $b \neq 0$ and $c = 0$, then we get $M(t) = a - \frac{t}{b} M'(t)$, $\forall\ t \in (-\infty, \infty)$. Hence $M(0) = a = 1$, which is not possible because $0 < a < 1$. Hence $c \neq 0$.

iii) If $b, c > 0$, then the differential equation (3.1) can be expressed as

$$M(t) - a = \frac{(c-t)}{b} \frac{dM(t)}{dt}$$

and we see that

$$\frac{dM(t)}{M(t)-a} = b \frac{dt}{(c-t)}.$$

On integrating both sides with respect to t, we obtain
$\log\,(M(t) - a) = -b \log(c - t) + k$.
That is
$M(t) - a = (c - t)^{-b} e^{k}$, where k is an arbitrary constant.
Therefore the solution of the differential equation (3.1) becomes

$$M(t) = a + (c - t)^{-b} e^{k}. \tag{3.3}$$

Since $M(0) = 1$, we get $e^{k} = (1 - a)c^{b}$.
From the equation (3.3),

$$M(t) = a + \frac{(1-a)}{(1-\frac{t}{c})^{b}}, \tag{3.4}$$

which is the mgf of zero-inflated gamma distribution with $a = \varphi$, $b = \alpha$, $c = \beta$, and $t < \beta$. This completes the proof of the theorem.

Also, it can be noted that when $b = \alpha = 1$, zero-inflated gamma distribution reduces to zero-inflated exponential distribution.

4. CONCLUSION

The result of this paper is purely of theoretical interest. The consequences and applications of this characterization are yet to be explored. But the relevance of zero-inflated gamma model is outlined in the following situation.

Consider a queueing system where the waiting time of a customer is gamma distributed and the purpose of the service is to collect a filled in form and issuing an acknowledgement. If there is no queue when a customer arrives at the service counter, the service will be completed instantaneously. Then, zero-inflated gamma distribution becomes an appropriate model for the waiting time of a customer in this queueing system.

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