Journal of Computer and Mathematical Sciences, *Vol.8(8), 359-365 August 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Fixed Point Theorems of Asymptotically Regular Mappings in Cone Metric Spaces

Dhirendra Kumar Singh

Govt. Vivekanand P.G. College,
Maihar, District, Satna (M.P.) INDIA.
email:dk.singhmyr@gmail.com

(Received on: August 8, Accepted: August 10, 2017)

ABSTRACT

The aim of this paper is to obtain the concept of asymptot-ically regular sequences and asymptotically regular maps in cone metric space and using this concept and without assuming normality, I establish unique fixed point theorems for self maps in a cone metric space.

Mathematics Subject Classification: 54H25, 47H10

Keywords: Fixed point, Cone metric space, asymptotically regular se-quences and maps.

1. INTRODUCTION

In 2007, Huang and Zhang[9] introduced the notion of cone metric spaces (CMSs) by replacing real numbers with an ordering Banachspace. The authors there gave an example of a function which is a contraction in the category of cone metric space but not contraction if considered over metric space and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. After that series of articles about cone metric spaces started to appear. Some of those articles dealt with the extension of certain fixed point theorems to cone metric spaces[2,3,7,10], and some other with the structure of the spaces themselves[2,11].

Very recently, Rezapour and Hamlbarani[10] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. On the other hand as long back Browder and Petryshgn[5] defined the asymptotically regular maps in metric space.

In the present paper, I define asymptotically regular sequences and maps with examples in cone metric spaces and the existence of fixed points for asymptotically regular maps on cone metric spaces is investigated. For the purpose of obtaining the fixed points of self maps we have used Hardy, Rogers condition (from[6]) and omitted the assumption of normality in cone metric space.

2. PRELMINARIES

Definition 2.1. [9]: Let B be a real Banach space and P be a subset of B. P is called a cone if.

(i) P is a closed,non empty and $P \neq \{0\}$
(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$.
(iii) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone P $\subseteq$ B, we define a partial ordering "$\leq$" in B by $x \leq y$ if $y-x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y-x \in P^0$ where P^0 stands for the interior of P.

Proposition 2.2[1]: Let P be a cone in a real Banach space B. If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = 0$.

Proposition 2.3[1]: Let P be a cone in a real Banach space B. If for $a \in B$ and $a \ll c$, for all $c \in P^0$, then $a = 0$

Remark 2.4[10]: $\lambda P^0 \subseteq P^0$, for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.

Definition 2.5[9]: Let X be a nonempty set. Suppose the mapping
$d : X \times X \rightarrow B$ satisfies:

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space (CMS).

Example 2.6([4][13]): Let $B = R^3$, $P = \{(x, y, z) \in B : x, y, z \geq 0\}$ and
$X = R$. Define $d : X \times X \rightarrow B$ by $d(x, y) = (\alpha \mid x - y \mid, \beta \mid x - y \mid, \gamma \mid x - y \mid)$ where α, β, γ are positive constants. Then (X, d) is a cone metric space.

Definition 2.7[9]: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in B$ with 0 _ c there is a positive integer Nc such that for all $n > N c$, $d(x_n, x)$ _ c, then the sequence $\{x_n\}$ is said to converges to x and x is called limit of $\{x_n\}$. We write $\lim_{n\rightarrow\infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 2.8[9]: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for any $c \in B$ with 0 _ c there is a N such that for all $n, m > N$, $d(x_n, x_m)$ _ c, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X.

Definition 2.9[9]: Let (X, d) be a cone metric space. If every Cauchy se-quence in X is convergent in X, then X is called a complete cone metric space.

Proposition 2.10[12]: Let (X, d) be a cone metric space and P be a cone in a real Banach space B. If $u \leq v$, v _ w then u _ w.

Lemma 2.11[12]: Let (X, d) be a cone metric space and P be a cone in a real Banach space B and k_1, k_2, k_3, k_4, $k > 0$.If $x_n \to x$, $y_n \to y$, $z_n \to z$, and $p_n \to p$ in X and
(i) $ka \leq k_1 d(x_n, x) + k_2 d(y_n, y) + k_3 d(z_n, z) + k_4 d(p_n, p)$ then $a = 0$.

Asymptotically regular sequences and maps: Here we will define asymp-totically regular sequences and maps in CMSs.

Definition 2.11: Let (X, d) be a cone metric space.A sequence $\{x_n\}$ in X is said to be asymptotically T- regular if $\lim_{n\to\infty} d(x_n, T x_n) = 0$.

Example 2.12(see[8]): Let $B = [0, 1]\times[0,1]$,$P=\{(x,y)\in B : x, y \geq 0\}$ and $X = [0, 1]$ define $d : X \times X \to B$ by $d(x, y) = (| x - y |, \alpha | x - y |)$ where α is positive constant. Then (X, d) is a cone metric space. Let T be a self map of X such that $T x = x/2$ and choosing a sequence
$\{x_n\}$, $x_n = 0$ for any positive integer n, converging to zero. We deduce that.

$\lim_{n\to\infty} d(x_n, T x_n) = \lim_{n\to\infty} | x_n - T x_n |, \alpha | x_n - T x_n |$

$= \lim_{n\to\infty} (| x_n - \frac{x_n}{2}|, \alpha | x_n - \frac{x_n}{2}|)$
$= \lim_{n\to\infty} (|\frac{x_n}{2}|, \alpha | \frac{x_n}{2}|)$
$=(0,0) =0$
Hence,$\{ x_n \}$ is an asymptotically T-regular space in (X,d).

Definition 2.13: Let (X, d) be a cone metric space. A mapping T of X into itself is said to be asymptotically regular at a point x in X if

$$\lim_{n\to\infty} d (T^n x, T^{n+1} x) = 0$$

Where $T^n x$ denotes the n^{th} iterate of T at x.

Example 2.14: Let (X,d) be a cone metric space which is defined in Ex. 2.12 and let T: X $\to$ X s.t. $Tx={}^{x}/_{3}$
where $x \in X$, then we have
$$\lim_{n\to\infty} d (T^n x, T^{n+1} x) = \lim_{n\to\infty} (|T^n x-T^{n+1} x |, \alpha(|T^n x-T^{n+1} x |)$$
$= \lim_{n\to\infty} (|x/3^n - x / 3^{n+1} |, \alpha|x/3^n - x / 3^{n+1} |)$
$=(|0-0|, \alpha|0-0|)$
$=(0,0)=0.$
Hence T be an asymptotically regular map at all point of X .

3. MAIN RESULTS

Theorem 3.1. Let (X, d) be a complete cone metric space and T be a self mapping of X

satisfying the inequality.

$d(T x, T y) \le a_1 d(x, T x) + a_2 d(y, T y) + a_3 d(x, T y)$
$+a_4 (d(y, T x) + d(x, y))$ (3.1.1)

for all $x, y \in X$ where $a_1, a_2, a_3, a_4 \ge 0$ and

$\max \{ (a_1 + a_4), (a_3 + 2a_4) \} < 1$ (3.1.2)

If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point

Proof: Let $\{x_n\}$ be an asymptotically T-regular sequence in X. Then

$d(x_n, x_m) \le d(x_n, T x_n) + d(T x_n, x_m)$
$\le d(x_n, T x_n) + d(T x_n, T x_m) + d(T x_m, x_m)$
$\le d(x_n, T x_n) + d(T x_m, x_m) + a_1 d(x_n, T x_n) +$
$a_2 d(x_m, T x_m) + a_3 d(x_n, T x_m) + a_4 d(x_m, T x_n) + a_4 d(x_n, x_m)$
$\le d(x_n, T x_n) + d(T x_m, x_m) + a_1 d(x_n, T x_n) + a_2 d(x_m, T x_m)$
$+ a_3 d(x_n, x_m) + a_3 d(x_m, T x_m) + a_4 d(x_m, x_n) + a_4 d(x_n, T x_n)$
$+ a_4 d(x_n, x_m)$
$= (1+a_1 + a_4) d(x_n, T x_n) + (1+a_2 + a_3) d(x_m, T x_m) + (a_3 + 2a_4) d(x_n, x_m)$
$\Rightarrow [1 - (a_3 + 2a_4)] d(x_n, x_m) \le (1 + a_1 + a_4) d(x_n, T x_n)$
$+ (1 + a_2 + a_3) d(x_m, T x_m)$
$\Rightarrow d(x_n, x_m) \le \frac{(1 + a_1 + a_4)}{[1-(a_3+2a_4)]} d(x_n, T x_n) + \frac{(1 + a_2 + a_3)}{|1-(a_3+2a_4)|} d(x_m, T x_m)$

$d(x_n, x_m) \le k_1 d(x_n, T x_n) + k_2 d(x_m, T x_m)$

Where $k_1 = 1 + a_1 + a_4/[1 - (a_3 + 2a_4)] > 0$ and

$k_2 = 1 + a_2 + a_3/[1 - (a_3 + 2a_4)] > 0$

Since $\{x_n\}$ is an asymptotically T-regular sequence and $m > n$.

Therefore $d(x_n, T x_n) = 0$ and $d(x_m, T x_m) = 0$ when $n \to \infty$

Let $0 << c$ be given and choose a natural number N, such that.

$[k_1 d(x_n, T x_n) + k_2 d(x_m, T x_m)] << c$ for all $m, n \ge N$,

Thus. $d(x_n, x_m) << c$ for $m > n$. Therefore $\{x_n\}$ is a Cauchy sequence in X which is a complete.

So $\{x_n\} \to x \in X$.

Existence of Fixed Point: Consider

$d(T x, x) \le d(T x, T x_n) + d(T x_n, x_n) + d(x_n, x)$

$\le a_1 d(x, T x) + a_2 d(x_n, T x_n) + a_3 d(x, T x_n) + a_4 d(x_n, T x) + a_4 d(x, x_n) + d(T x_n, x_n) + d(x_n, x)$

$\le a_1 d(x, T x) + a_2 d(x_n, T x_n) + a_3 d(x, x_n) + a_3 d(x_n, T x_n) + a_4 d(x_n, x) + a_4 d(x, T x) + a_4 d(x, x_n)$
$+ d(T x_n, x_n) + d(x_n, x)$

$= (a_1 + a_4) d(x, T x) + (1 + a_2 + a_3) d(x_n, T x_n) + (1 + a_3 + 2a_4) d(x, x_n)$

$\Rightarrow d(x, T x) \le \frac{(1 + a_2 + a_3)}{(1-a_1 - a_4)} d(x_n, T x_n) + \frac{(1 + a_3 + 2a_4)}{(1-a_1 - a_4)} d(x, x_n)$

Since $\{x_n\}$ is an asymptotically T- regular sequence and
$x_n \to x$ implies that $d(x_n, T x_n) = 0$ and $d(x, x_n) \to 0$
as $n \to \infty$, using fact $a_1 + a_4 < 1$ and Lemma 2.11, we have $d(T x, x) = 0 \Rightarrow T x = x$

Uniqueness: Let z be another fixed point of T then

$d(x, z) = d(T x, T z)$

$\leq a_1 d(x, T x) + a_2 d(z, T z) + a_3 d(x, T z) + a_4 d(z, T x) + a_4 d(x, z)$

$= (a_3 + 2a_4) d(x, z)$

$\Rightarrow \;\; [1 - (a_3 + 2 a_4)] d(x, z) \leq 0$

$\Rightarrow \;\; d(x, z) = 0$ [by Prop. 2.2 and $(a_3 + 2 a_4) < 1$]

$\Rightarrow \;\; x = z$

This completes the proof.

Theorem 3.2. Let (X, d) be a complete cone metric space and T be a contin-uous self mapping of X . If there exists an asymptotically T- regular sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x$ then x is a fixed point of T .

Proof: Now $d(T x, x) \leq d(T x, T x_n) + d(T x_n, x_n) + d(x_n, x)$ Then taking limit as $n \to \infty$, we have $d(T x, x) = 0$ [using Lemma 2.11]

$\Rightarrow T x = x$
Hence x is a fixed point of T .

Theorem 3.3. Let (X, d) be a complete cone metric space and T a self mapping of X, satisfying the condition.

$$d(T x, T y) \leq a_1 d(x, T x) + a_2 d(y, T y) + a_3 d(x, T y) + a_4 d(y, T x) + a_4 d(x, y) \tag{3.1.3}$$

for all $x, y \in X$
where $a_1, a_2, a_3, a_4 \geq 0$ $\max\{(a_1 + a_4), (a_3 + 2a_4)\} < 1$ (3.1.4)

If T is an asymptotically regular at some fixed point x of X then there exists a unique fixed point of T .

Proof: Let T be an asymptotically regular at $x_0 \in X$. Consider the sequence $\{T^n x_0\}$ then for all $m, n \geq 1$
$$d(T^m x_0, T^n x_0) \leq a_1 d(T^{m-1} x_0, T^m x_0) + a_2 d(T^{n-1} x_0, T^n x_0) + a_3 d(T^{m-1} x_0, T^n x_0) + a_4 d(T^{n-1} x_0, T^m x_0) + a_4 d(T^{m-1} x_0, T^{n-1} x_0)$$

$\leq a_1 d(T^{m-1}x_0, T^m x_0) + a_2 d(T^{n-1}x_0, T^n x_0)$
$+ a_3 d(T^{m-1}x_0, T^m x_0) + a_3 d(T^m x_0, T^n x_0)$
$+ a_4 d(T^{n-1}x_0, T^n x_0) + a_4 d(T^n x_0, T^m x_0)$

$+ a_4 d(T^{m-1}x_0, T^m x_0) + a_4 d(T^m x_0, T^n x_0)$
$+ a_4 d(T^n x_0, T^{n-1}x_0)$

$\Rightarrow d(T^m x_0, T^n x_0) \leq \frac{(a_1 + a_3 + a_4)}{[1-(a_3+2a_4)]} d(T^{m-1}x_0, T^m x_0) + \frac{(a_2 + 2a_4)}{[1-(a_3+2a_4)]} d(T^{n-1}x_0, T^n x_0)$

$\Rightarrow d(T^m x_0, T^n x_0) \leq k_1 d(T^{m-1}x_0, T^m x_0) + k_2 d(T^{n-1}x_0, T^n x_0)$

Where $k_1 = (a_1 + a_3 + a_4)/1 - (a_3 + 2a_4) > 0$

and $k_2 = (a_2 + 2a_4)/1 - (a_3 + 2a_4) > 0$ [By 3.1.4]

Since T is an asymptotically regular at x_0, therefore

$d(T^{m-1}x_0, T^m x_0) = 0$ and $d(T^{n-1}x_0, T^n x_0) = 0$

as $m, n \to \infty$

Let $0 \ll c$ be given and choose a natural number N such that

$k_1 d(T^{m-1}x_0, T^m x_0) + k_2 d(T^{n-1}x_0, T^n x_0) \ll c$

for all $m, n \geq N$ Thus $d(x_n, x_m) \ll\_ c$ for $m > n$

Therefore $\{T^n x_o\}$ is a Cauchy sequence in X. Which is complete so $\{T^n x_o\} \to x \in X$. Now we claim that x is a fixed point of T. For this

$d(Tx, x) \leq d(Tx, T^n x_0) + d(T^n x_0, x)$

$\leq a_1 d(x, Tx) + a_2 d(T^{n-1}x_0, T^n x_0) + a_3 d(x, T^n x_0)$

$+ a_4 d(T^{n-1}x_0, Tx) + a_4 d(x, T^{n-1}x_0) + a_4 d(T^n x_0, x)$

$\leq a_1 d(x, Tx) + a_2 d(T^{n-1}x_0, T^n x_0) + a_3 d(x, T^n x_0)$

$+ a_4 d(T^{n-1}x_0, T^n x_0) + a_4 d(T^n x_0, Tx) + a_4 d(x, T^n x_0)$

$+ a_4 d(T^n x_0, T^{n-1}x_0) + d(T^n x_0, x)$

$\Rightarrow d(Tx, x) \leq (a_1 + a_4) d(Tx, x)$ (when $n \to \infty$)

$\Rightarrow [1 - (a_1 + a_4)] d(Tx, x) \leq 0$ [by Prop 2.2 and

as $a_1 + a_4 < 1$]

$\Rightarrow Tx = x$

The Unique fixed point x follows as in theorem 3.1.

Hence, Theorem is complete.

ACKNOWLEDGEMENT

The author is thankful to J.O. Olaleru for their valuable suggestions which improved the presentation of research paper.

REFERENCES

1. D. Ilic, V. Rakocevie, Quasi contraction on a cone metric space, *Applied Mathematics Letters*, article in press.
2. D. Turkoglu and M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, *Acta Mathematica Sinica*, *English Series*, submitted.

3. D. Turkoglu and M. Abuloha and T. Abdeljawad, KKM mappings in cone metric spaces and some fixed point theorems, *Nonlinear Analysis: Theory, Methods and Applications*, Vol.72, No. 1, PP 348-353, (2010).
4. Erdal Karapinar Fixed point Theorems in cone Banach spaces, *Fixed Point Theory and Applications* Vol 2009, Article ID 609281.
5. F.E. Browder, W.V. Petryshyn,. The solution by iteration of nonlinear functional equations in Banach spaces. *Bull. Amer Math. Soc*., 72, 571-575 (1966).
6. Hardy G. E. and Rogers T.D., A generalization of fixed point theorem of *Reich Canad, Math. Bull*. 16, 201 - 206, (1973).
7. I. Sahin and M. Telci, Fixed Points of contractive mappings on complete cone metric spaces, *Hacettepe Journal of Mathematics and Statistics*, Vol 38, no.1 PP. 59-67, (2009).
8. J.O. Olaleru, Some generalizations of fixed point theorems in cone metric spaces, *Fixed Point Theory and Applications* Vol. 2009, Article ID 657914.
9. L. G. Huang and X. Zhang, Cone metric Spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications,* Vol. 332, No. 2, PP 1468 - 1476, (2007).
10. Sh. Rezapour and R. Hamlbarani, Some notes on the Paper Cone met-ric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications*, Vol. 345, No. 2, PP 719 - 724, (2008).
11. Sh. Rezapour, M. Derafshpour, and R. Hamlbarani, A Review on Topo-logical properties of cone metric spaces, in Analysis Topology and Applica-tions 2008 (ATA 2008), Technical Faculty Cacak University of Kragujevac Vrnjacka Banja, Serbia, the 30th of May to the 4th of June, (2008).
12. Shobha Jain, Shishir Jain, Lal Bahadur, Compatibility and weak compat-ibility for four self maps in a cone metric space. *Bulletin of Mathematical Analysis and Applications*, Vol 2, Issue 1, Pages 15 – 24 (2010).
13. Thabet Abdeljawad and Erdal Karapinar Quasi cone metric spaces and Generalization of Caristi Kirk’s theorem, *Fixed Point Theory and Applications*, Vol 2009, Article ID 574387.