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Fixed Point Theorems for Nonexpansive Mappings of Banach Space

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ABSTRACT

Suppose that X be a uniformly convex Banach space, let E be nonempty subset of P with P as a nonexpansive retraction. The iterative scheme is defined by (2.1), we establish fixed point theorem for weak convergence for nonexpansive mappings of Banach space.

Keywords: Nonexpansive mapping, Banach space, Fixed point and Retraction.

1. INTRODUCTION

Let E be a closed convex bounded subset of a Banach space X and T:E→E be a mapping Then T is called nonexpansive mappings

if $\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in E$ (1.1)

Let F(T) = {x∈F:T(x) = x}, then F(T) is called the set of fixed points of a mapping T.

The first nonlinear ergodic theorem was proved by Baillon[5] for general nonexpansive mappings in Hilbert space H: if k is a closed and convex subset of H ant T has a fixed point then ∀ x ∈ K, $\{T_n(x)\}$ is weakly convergent, as n→∞, to a fixed point of T. It was also shown by Pazy[1] that if H is a real Hilbert space and $\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)$ converges weakly to y ∈ E, as n→∞ then y ∈F(T). Rhoades and Temir[2] introduced the concept of a quasi nonexpansive mapping was initiated Diaz and Metcaff[6] and Dotson[12] studied quasi nonexpansive mapping in Banach spaces. Recently, this concept was given by Kirk[11] in metric spaced which we about to a normed space as follows. The mapping T is called a quasi nonexpansive mapping if

$\|T(x) - f\| \leq \|x - f\|, \qquad \forall x \in E \; f x \in F(T).$ (1.2)

2. PRELIMINARIES AND DEFINITIONS

Let X be a real Banach space. A subset E of X is said to be a retract of X if ∃ a continuous map $P:X \to E$ such that

$Px = x, \forall x \in E$,

A map $P:E \to E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction, then Py = y for all y in the range of P.

In[2], Rhoades and Temir considered T and I self mapping of X, where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I. They prove the following theorem.

Theorem (Rhoades and Temir [2])

Let E be a closed convex bounded subset of uniformly convex Banach space X, which satisfies Opial's condition and let T, I self mappings of E with T be an I-nonexpansive mapping, the sequence $\{x_n\}$ of Ishikawa iterates converges weakly to common fixed point of $F(T) \cap F(I)$.

Let X be a Banach space and let E be a nonempty convex subset of X.

Let $T,S:E \to E$ be two given mappings and $P:E \to E$ is a retraction. The iteration scheme $\{x_n\}$ is defined by $x_0 = x \in E$ and

$$x_{n+1} = P\{\alpha_n T(x_n) + \beta_n S(y_n) + \gamma_n T(S(x_n))\} \tag{2.1}$$

$$y_n = P(\alpha'_n T(x_n) + \beta'_n S(y_n) + \gamma'_n x_n)$$

where $\alpha_n + \beta_n + \gamma_n = 1$

$\alpha'_n + \beta'_n + \gamma'_n = 1$

and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in (0, 1).

A Banach space X is said to satisfy Opial's condition [15], if for each seq. $\{x_n\}$ in X, $x_n \to x$ implies that

$$\lim_{n\to\infty} \|x_n - x\| < \lim_{n\to\infty} \|x_n - y\| \tag{2.2}$$

$\forall y \in X$ with $y \neq x$.

In this paper, I consider that T and S are nonexpansive mappings in a Banach space. I establish the weak convergence theorem of the sequence of iterative scheme (2.1) to a common fixed point of $F(T) \cap F(S)$.

3. MAIN RESULT

3.1 Theorem

Let E be a closed bounded subset of a uniformly convex Banach space X, which satisfies Opial's condition and let T, S be self mappings of E. T and S are nonexpansive mappings on E. Then for $x_0 \in E$, the sequence $\{x_n\}$ of iteration scheme defined by (2.1) converges weakly to common fixed point of $F(T) \cap F(S)$.

Proof

Let F(T)∩F(S) be nonempty and a singleton, then the proof is obvious. So we assume that F(T)∩F(S) is nonempty and F(T)∩F(S) is not a singleton.

$$\|x_{n+1} - f\| = \left\|P\{\alpha_n T(x_n) + \beta_n S(y_n) + \gamma_n T\big(S(x_n)\big)\} - f\right\|$$
$$= \|\alpha_n T(x_n) + \beta_n S(y_n) + \gamma_n T(S(x_n)) - (\alpha_n + \beta_n + \gamma_n) f\|$$
$$\leq \alpha_n \|T(x_n) - f\| + \beta_n \|S(y_n) - f\| + \gamma_n \| T(S(x_n)) - f\|$$
$$\leq \alpha_n \|x_n - f\| + \beta_n \|y_n - f\| + \gamma_n \|S(x_n) - f\|$$
$$\leq \alpha_n \|x_n - f\| + \beta_n \|y_n - f\| + \gamma_n \|x_n - f\|$$
$$= (\alpha_n + \gamma_n)\|x_n - f\| + \beta_n \left\|P\big(\alpha'_n T(x_n) + \beta'_n S(x_n) + \gamma'_n x_n\big) - f\right\|$$
$$= (\alpha_n + \gamma_n)\|x_n - f\| + \beta_n \left\|\alpha'_n T(x_n) + \beta'_n S(x_n) + \gamma'_n x_n - (\alpha'_n + \beta'_n + \gamma'_n) f\right\|$$
$$\leq (\alpha_n + \gamma_n)\|x_n - f\| + \beta_n [\alpha'_n \|T(x_n) - f\| + \beta'_n \|S(x_n) - f\| + \gamma'_n \|x_n - f\|]$$
$$\leq (\alpha_n + \beta_n)\|x_n - f\| + \beta_n [\alpha'_n \|x_n - f\| + \beta'_n \|x_n - f\| + \gamma'_n \|x_n - f\|]$$
$$= (\alpha_n + \beta_n)\|x_n - f\| + \beta_n \big[(\alpha'_n + \beta'_n + \gamma'_n)\|x_n - f\|\big]$$
$$= (\alpha_n + \beta_n)\|x_n - f\| + \beta_n \|x_n - f\|$$
$$= (\alpha_n + \beta_n + \gamma_n)\|x_n - f\| = \|x_n - f\|$$

$\therefore \|x_{n+1} - f\| \leq \|x_n - f\|$

$\therefore \{\|x_n - f\|\}$ is a non increasing sequence

Then $lim_{n\to\infty} \|x_n - f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and S. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in E. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converges weakly to f and q respectively we shall show that f = q suppose that X statistics Opial condition and that f ≠ q is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \to f$ and $\{x_{m_k}\} \to q$ respectively. Since $\lim_{n\to\infty} \|x_n - f\|$ exists for any f ∈ F(T)∩F(S). By Opial's condition, we find that $\lim_{n\to\infty} \|x_n - f\| = \lim_{k\to\infty} \left\| x_{n_k} - f \right\|$

$$< \lim_{k\to\infty} \left\| x_{n_k} - q \right\| = \lim_{n\to\infty} \|x_n - q\|$$
$$< \lim_{j\to\infty} \left\| x_{m_j} - q \right\| = \lim_{j\to\infty} \left\| x_{m_j} - f \right\|$$
$$= \lim_{n\to\infty} \|x_n - f\|$$

This is a contradiction. This $\{x_n\}$ converges weakly to an element of F(T)∩F(S).

3.2 Theorem

Let E be a closed bounded subset of a uniformly convex Banach space X, which satisfies Opital's condition and let T. S be self mappings of E. T and S are quasi-nonexpansive mappings on E. Then for x_0 ∈ E, the sequence $\{x_n\}$ of iterative scheme defined by (2.1) converges weakly to common fixed point of F(T) ∩ F(S).

Proof: Every nonexpansive mappings is a quasi-nonexpansive mapping. The proof is similar to theorem (3.1).

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