Journal of Computer and Mathematical Sciences, *Vol.9(5), 373-383 May 2018* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Multifunction with Topological Closed Graphs

D. Sheeba and N. Nagaveni

Department of Mathematics,
Coimbatore Institute of Technology, Coimbatore-14, Tamil Nadu, INDIA.
email: sheebarinald@gmail.com, nagavenipalanisamy@yahoo.com

(Received on: May 1, 2018)

ABSTRACT

In this paper, we introduced and studied some basic properties of weaker form of multifunction such as upper and lower m_{wg}-continuous multifunction. We obtain some of its characterizations with totally m_{wg}- closed graph and strongly m_{wg}-closed graph in Minimal Structures.

AMS Subject Classification: 54C08, 54C60.

Keywords: Strongly m_{wg}- closed graph, totally m_{wg}- closed graph, m_{wg}- compact, m_{wg}- connected and etc.

1. INTRODUCTION

In 1975, Smithson[13] studied the concept of sub continuity is extended to multifunction and also he obtained a number of results on multifunction with closed graphs. In 1996, Cao and Reilly[1] introduced weaker form of multifunction such as α-continuous and α-irresolute multifunction in topological space.

In 2000, Noiri and Popa[6] defined and studied some properties of upper/lower M-continuous multifunction in Minimal structure. Popa *et al.*, investigates different types of generalization of multifunction such as almost continuous multifunction[11], almost nearly m-continuous multifunction[8], slightly m-continuous multifunction[7], almost weakly continuous multifunction[5] and etc. Nagaveni *et al.*, defined weakly generalized closed sets[9] and mg-continuous functions[10] in Minimal structures.

The purpose of this paper is to give a new weaker form of multifunction such as upper and lower m_{wg}-continuous multifunction are investigated in minimal structures. We obtain some of its characterizations with graph of multifunction, totally m_{wg}- closed graph and strongly m_{wg}- closed graph in Minimal Structures.

A multifunction $F: X \to Y$ is a point to set correspondence and we always assume that $F(x) \neq \Phi$ for every point x ∈ X. For a multifunction F, the upper and lower set V of Y will be denoted by F^+(V) and F^-(V) respectively, that is, F^+(V) = {x ∈ X: $F(x) \subset$ V} and F^-(V) = { x ∈ X: $F(x) \cap$ V ≠ Φ}. In particular, F^-(y) = {x ∈ X:$y \in F(x)$} for each point y ∈ Y. For each A ⊂ X, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$ or equivalently, if for each y ∈ Y there exists a x ∈ X such that $y \in F(x)$. The graph multifunction $G_F: (X, \tau) \to (X \times Y, \tau \times \sigma)$ of F is defined by $G_F(x) = \{\{x\} \times F(x)\}$ for each x ∈ X. Graph of F (ie.) G(F) = {(x, y) / x ∈ X, y ∈ F(x)}. We say that F has a closed graph if G(F) is closed in (X × Y, τ × α). Throughout the paper (X, m_X) and (Y, m_Y) are denoted by minimal structure (briefly. m-space). The interior and closure of a subset A of (X, m_X) are denoted by m_X -Int(A) and m_X -Cl(A) respectively.

2. PRELIMINARIES

In this section, we list some definitions which are used in this sequel.

Definition: 2.1 [6] Let X be a non empty set and P(X) the power set of X. A subfamily m_X of P(X) is called a minimal structure (briefly m-structure) on X if Φ ∈ m_X and X ∈ m_X.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be m_X -open and the complement of an m_X -open set is said to be m_X -closed.

Definition: 2.2 [6] An m-structure m_X on a nonempty set X is said to have property B if the the union of any family of subsets belong to m_X belongs to m_X.

Definition: 2.3 [6] Let X be a nonempty set and m_X an m-structure on X. For subset A of X, the m_X-closure of A and the m_X -interior of A are defined in as follows:

i. m_X -Cl(A) = ∩{F : A⊂ F, X - F ∈ m_X},
ii. m_X -Int(A) = ∪{U : U ⊂ A, U ∈ m_X}.

Lemma: 2.4 [6] Let (X, m_X) be a space with minimal structure, let A be a subset of X and x ∈ X. Then x ∈ m_X -Cl(A) if and only if U ∩ A ≠ Φ, for every U ∈ m_X containing the point x.

Lemma: 2.5 [6] Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following hold:

i. m_X -Cl(X - A) = X - (m_X -Int(A)) and m_X -Int(X - A) = X - (m_X -Cl(A)),
ii. If (X - A) ∈ m_X, then m_X -Cl(A) = A and if A ∈ m_X, then m_X -Int(A) = A,
iii. m_X -Cl(Φ) = Φ, m_X -Cl(X) = X, m_X -Int(Φ) = Φ and m_X -Int(X) = X,
iv. If A ⊂ B, then m_X -Cl(A) ⊂ m_X -Cl(B) and m_X - Int(A) ⊂ m_X - Int(B),
v. A ⊂ m_X -Cl(A) and m_X -Int(A) ⊂ A,
vi. m_X -Cl(m_X -Cl(A)) = m_X -Cl(A) and m_X -Int(m_X -Int(A)) = m_X -Int(A).

Definition: 2.6 [9] A subset A of a m-space (X, m_X) is said to be minimal weakly generalized closed (briefly. mwg-closed) sets if m_X - Cl (m_X - Int(A)) ⊂ U whenever A ⊂ U and U is open in m_X.
The complement of mwg-closed set is said to be mwg-open set. The family of all mwg-open (resp. mwg-closed) sets is denoted by m_X -WGO(X) (resp. m_X -WGC(X)). We define, m_X - WGO(X, x) = {V ∈ m_X -WGO(X) / x ∈ V} for x ∈ m_X .

Lemma: 2.7 [6] For a multifunction $F: (X, m_X) \to (Y, m_Y)$ following hold:
(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$, for any subsets A ⊆ X and B ⊆ Y.

Definition: 2.8 [2] A m-space(X, m_X) is said to be
i. m- Haussdroff if for any distinct points x, y there exists U, V ∈ m_X such that x ∈ U, y ∈ V and U ∩ V = Φ.
ii. m-Urysohn if for any distinct points x, y there exists U, V ∈ m_X such that x ∈ U, y ∈ V and m_X -Cl(U) ∩ m_X -Cl(V) = Φ.
iii. m-compact if every cover of X by mwg-open sets has a finite subcover.

Definition: 2.9 [12] A m-space (X, m_X) is called
i. m_{wg}-Haussdroff space (i.e. m_{wg}-T_2 space) if for every pair of distinct points x, y in X there exists disjoint mwg-open sets U ∈ X and V ∈ X containing x and y respectively.
ii. mwg-normal if for each pair of non empty disjoint m-closed sets can be separated by disjoint mwg-open sets.
iii. mwg-regular if for each mwg-closed set F of X and each x ∉ F, there exist disjoint mwg-open sets U and V such that F ⊂ U and x ∈ V.

Definition: 2.10 [3] A function f: (X, τ) → (Y, σ) is said to be
i. upper semi continuous, (or u.s.c) if F^+(V) is open in (X, τ) for each open set V of (Y, σ).
ii. upper α-continuous, (or u.α.c) if F^+(V) is α-open in (X, τ) for each open set V of (Y, σ).
iii. lower semi continuous, (or l.s.c) if F^-(V) is open in (X, τ) for each open set V of (Y, σ).
iv. lower α-continuous, (or l.α.c) if F^-(V) is α-open in (X, τ) for each open set V of (Y, σ).

Remark: 2.11[6] Let m_X = α(X) (resp. SO(X)) and m_Y = σ. Then upper / lower M continuous multifunction $F: (X, m_X) \to (Y, \sigma)$ is an upper / lower α continuous (resp. an upper / lower semi continuous.

Remark: 2.12 [6] A multifunction $F: (X, \alpha(X)) \to (Y, \sigma)$ is an upper / lower α continuous, if it is upper / lower semi continuous.

Remark: 2.13 [4] If a subset A of X is α-closed, then it is weakly generalized closed (abbreviated as wg-closed) set.

3. UPPER AND LOWER m_{wg}-CONTINUOUS MULTIFUNCTION

In this section, we defined and investigated a new weaker form of multifunction such as upper and lower m_{wg}-continuous multifunction in Minimal Structures.

Definition: 3.1 A multifunction $F: (X, m_X) \to (Y, m_Y)$ is called
(i). upper m_{wg}-continuous (briefly, u. m_{wg}-c.) at a point x ∈ X if for each m-open subset V of Y with $F(x) \subseteq$ V, there is an mwg-open set U containing x such that $F(U) \subseteq$ V.
(ii). lower m_{wg}-continuous (briefly, l. m_{wg}-c) at a point x ∈ X if for each m-open subset V of Y with $F(x) \cap$ V ≠ Φ, there is an mwg-open set U containing x such that $F(y) \cap$ V ≠ Φ, for every point y ∈ U.

Remark: 3.2 From the following examples, it is clear that upper m_{wg}-continuous and lower m_{wg}-continuous are independent of each other.

Example: 3.3 Let X = {p, q, r} and Y = {a, b, c} be endowed with the minimal structures m_X = {X, Φ, {p}} and m_Y = {Y, Φ, {b}, {c}, {b, c}}. If multifunction $F: (X, m_X) \to (Y, m_Y)$ is defined by $F(x) = \begin{cases} \{a\} & if\ x = p \\ Y & if\ x = q, r \end{cases}$. Then F is upper m_{wg}-continuous, but it is not lower m_{wg}-continuous.

Example: 3.4 Let X = {a, b, c, d} and Y = {p, q, r} be endowed with the minimal structures m_X = {X, Φ, {a}, {b, c}, {a, b, c}} and m_Y = {Y, Φ, {p, q}}. If multifunction $F: (X, m_X) \to (Y, m_Y)$ is defined by $F(x) = \begin{cases} \{p\} & if x = a \\ Y & if\ x = b\ or\ c \\ \{p, q\} & if\ x = d \end{cases}$. Then F is lower m_{wg}-continuous, but it is not upper m_{wg}-continuous.

Theorem: 3.5 Let $F: (X, m_X) \to (Y, m_Y)$ be a multifunction. Then the following statements are equivalent.
(i) $F: (X, m_X) \to (Y, m_Y)$ is an upper m_{wg}-continuous.
(ii) F^+(V) ∈ m_X- WGO(X) for each V ∈ m_X- O(Y).
(iii) F^-(V) ∈ m_X- WGC(X) for each V ∈ m_X- O(Y).

Proof: (i) ⇔ (ii) Let V be a m-open subset set of m_Y and x ∈ F^+(V). Since $F: (X, m_X) \to (Y, m_Y)$ is an upper m_{wg}-continuous, there exists U ∈ m_X- WGO(X, x) such that $F(U) \subseteq$ V. Hence, F^+(V) is mwg-open in X.
(ii) ⇔ (iii) It follows from the fact that F^+(Y\V) = X \ F^-(V) for every subset V of Y.

Theorem: 3.6 Let $F: (X, m_X) \to (Y, m_Y)$ be a multifunction. Then the following statements are equivalent.
(i) $F: (X, m_X) \to (Y, m_Y)$ is a lower m_{wg}-continuous.

(ii) F^{-}(V) $\in m_X$- WGO(X) for each V $\in$ m_X- O(Y).
(iii) F^{+}(V) $\in m_X$- WGC(X) for each V $\in$ m_X- O(Y).
The proof follows from the definitions and properties.

Theorem: 3.7 Let $F: (X,\ m_X) \to (Y,\ m_Y)$ be a multifunction. Then the following statements are equivalent.
(i) $F: (X,\ m_X) \to (Y,\ m_Y)$ is an upper m_{wg}-continuous.
(ii) F^{+}(V) = m_X -Int(F^{+}(V)) for each V $\in$ m_Y.
(iii) F^{-}(V) = m_X -Cl(F^{-}(V)) for each m-closed set of m_Y.

Proof: (i) ⇒ (ii): let V be any m-open set of m_Y and x ∈ F^{+}(V). Then, F(x) ∈ V. There exist mwg-open set U ∈ m_X containing x such that F(U) ⊂ V. Thus x ∈ U ⊂ F^{+}(V) and hence x ∈ m_X -Int(F^{+}(V)). Therefore, we have F^{+}(V) ⊂ m_X -Int(F^{+}(V)). By Lemma 2.5, we have m_X -Int(F^{+}(V)) ⊂ F^{-}(V). Therefore, we obtain F^{+}(V) = m_X -Int(F^{+}(V)).
(ii) ⇒ (iii): Let V be any m-closed set in Y. Then Y - V is m-open in m_Y. By (ii) and Lemma 2.5, we have X - F^{-}(V) = F^{+}(Y - V) = m_X -Int(F^{+}(Y - V)) = m_X -Int(X - F^{-}(V)) = X - m_X -Cl(F^{-}(V)). Therefore, we obtain F^{-}(V) = m_X -Cl(F^{-}(V)) .
(iii) ⇒ (ii): This follows from the fact that F^{-}(Y - B) = X - F^{+}(B) for every subset B of Y.
(ii) ⇒ (i): Let x ∈ X and V be any m-open set of Y containing F(x). Then x ∈ F^{+}(V) = m_X -Int(F^{+}(V)). There exist mwg-open set U ∈ m_X containing x such that x ∈ U ⊂ F^{+}(V). Therefore we have x ∈ U ⊂ F^{+}(V). Therefore, we have x ∈ U, U ∈ m_X and F(U) ⊂ V. Hence F is upper m_{wg}-continuous.

Theorem: 3.8 Let $F: (X,\ m_X) \to (Y,\ m_Y)$ be a multifunction. Then the following statements are equivalent.
(i) $F: (X,\ m_X) \to (Y,\ m_Y)$ is lower m_{wg}-continuous.
(ii) F^{-}(V) = m_X -Int(F^{-}(V)) for each V $\in$ $m_{Y.}$
(iii) F^{+}(V) = m_X -Cl(F^{+}(V)) for each m-closed set of m_Y.

Proof: (i) ⇒ (ii): let V be any m-open set of m_Y and x ∈ F^{-}(V). Then, F(x) ∩ V ≠ Φ. By (i), there exist mwg-open set U ∈ m_X containing x such that F(u) ∩ V ≠ Φ for u ∈ U. Therefore we have U ⊂ F^{-}(V) and hence x ∈ U ⊂ m_X -Int(F^{-}(V)). Thus, we obtain F^{-}(V) ⊂ m_X -Int(F^{-}(V)) and by the Lemma 2.5, F^{-}(V) = m_X -Int(F^{-}(V)).
(ii) ⇒ (iii): Let V be any m-closed set in Y. Then Y - V is m-open in m_Y. By (ii) and Lemma 2.5, we have X - F^{+}(V) = F^{-}(Y - V) = m_X -Int(F^{-}(Y - V)) = X - m_X -Cl(F^{+}(V)). Hence we obtain F^{+}(V) = m_X -Cl(F^{+}(V)).
(iii) ⇒ (ii): Let x ∈ X and V be any m-open set of Y containing F(x) ∩ V ≠ Φ. Then x ∈ F^{-}(V) and x ∉ X - F^{-}(V) = F^{+}(Y - V). By (iii) we have x ∉ m_X -Cl(F^{+}(Y - V)). By Lemma 2.4, there exist U ∈ m_X containing x such that U ∩ F^{+}(Y - V) = Φ. Hence U ⊂ F^{-}(V). Therefore, F(u) ∩ V ≠ Φ for each u ∈ U and F is lower m_{wg}-continuous.

Definition: 3.9 A topological space (X, τ) is said to be m-extremely disconnected if the closure of each m-open set of X is m-open in X.

Theorem: 3.10 Let (Y, m_Y) be m-extremely disconnected. For a multifunction $F: (X, m_X) \rightarrow (Y, m_Y)$ following statements are equivalent.
(i) $F: (X, \tau) \rightarrow (Y, m_Y)$ is upper m_{wg}-continuous.
(ii) m_X -Cl(F^{-}(U)) ⊂ F^{-}(m_Y -Cl(U)) for every m-open set U of m_Y.
(iii) F^{+}(m_Y -Int(V)) ⊂ m_X -Int(F^{+}(V)) for every m-closed set V of m_Y.

Proof: (i) ⇒ (ii) Let V be any m-open set of m_Y. Then m_Y-Cl(V) is m_Y-closed. By Theorem 3.7, we have F^{-}(U) ⊂ F^{-}(m_Y -Cl(U)) = m_X -Cl(F^{-}(m_Y -Cl(U))) and m_X -Cl(F^{-}(U)) ⊂ F^{-}(m_Y -Cl(U)).
(ii) ⇒ (iii) Let V be any m-closed set of m_Y. Let U = Y - V. Then U is m-open in m_Y. By Lemma 2.5, we have X - m_X -Int(F^{+}(V)) = m_X -Cl(X - F^{-}(V)) = m_X -Cl(F^{-}(Y - V)) ⊂ F^{-}(m_Y -Cl(Y - V)) = F^{-}($Y -$ m_Y -Int(V)) = X - F^{+}(m_Y -Int(V)). Therefore, we obtain F^{+}(m_Y -Int(V)) ⊂ m_X -Int(F^{+}(V)).
(iii) ⇒ (i) Let x ∈ X and V ∈ m_Y containing$F(x)$. Then by (iii) x ∈F^{+}(V) = F^{+}(m_Y -Int(V)) ⊂ m_X -Int(F^{+}(V)). There exist U ∈ m_X containing x such that U ⊂ (F^{+}(V)). Hence F(U) ⊂ V. This implies that $F: (X, \tau) \rightarrow (Y, \sigma)$) is upper m_{wg}-continuous.

Theorem: 3.11 Let (Y, m_Y) be m-extremally disconnected. For a multifunction $F: (X, m_X) \rightarrow (Y, m_Y)$ following statements are equivalent.
(i) $F: (X, m_X) \rightarrow (Y, m_Y)$ is lower m_{wg}-continuous.
(ii) For any m-open set B of m_Y, m_X -Cl(F^{+}(B)) ⊆ F^{+}(Cl(B),
(iii) For any m-closed C of Y, F^{-}(Int(C)) ⊆ m_X -Int($F^{-}(C)$),

Proof: The proof is obivious.

Theorem: 3.12 If a multifunction $F: (X, \alpha(X)) \rightarrow (Y, \sigma)$ is an upper / lower α continuous, then it is upper / lower m_{wg}-continuous multifunction.

Proof: Suppose that $F: (X, \alpha(X)) \rightarrow (Y, \sigma)$ is an upper / lower α continuous at x ∈ X. If V is an open set in Y with F(x) ⊂ V (resp. F(x) ∩ V ≠ Φ). Then, F^{+}(V) (resp.$F^{-}(V)$) is an α-open set in X. From the Remark 2.13, every α-open set is wg-open set. Thus F is an upper / lower α continuous at x ∈ X.

Remark: 3.13 If a multifunction F is an upper / lower semi continuous, then it is upper / lower m_{wg}-continuous multifunction.

Theorem: 3.14 If $F: (X, \alpha(X)) \rightarrow (Y, m_Y)$ is an upper α continuous (resp. lower α continuous) and $G: (Y, m_Y) \rightarrow (Z, m_Z)$ is upper m_{wg}-continuous (resp. lower m_{wg}-continuous), Then $G \circ F$ is upper m_{wg}-continuous (resp. lower m_{wg}-continuous).

Proof: Let V be a m-open set in Z. Since G is upper m_{wg}-continuous (resp. lower m_{wg}-continuous), G^{+}(V) (resp. $G^{-}(V)$) is an mwg-open in Y. Also since F is an upper α continuous (resp. lower α continuous), $F^{+}(G^{+}$(V)) = $(G \circ F)^{+}$(V) (resp. $F^{-}(G^{-}$(V)) = $(G \circ F)^{-}$(V) is an wmg-open set in X. Thus $G \circ F$ is upper m_{wg}-continuous (resp. lower m_{wg}-continuous).

4. CHARACTERIZATION OF UPPER AND LOWER m_{wg}-CONTINUOUS MULTIFUNCTION WITH GRAPH OF MULTIFUNCTION

We obtain some of upper and lower m_{wg}-continuous characterizations with graph of multifunction, totally m_{wg}- closed graph and strongly m_{wg}- closed graph in Minimal Structures. Recall that, if F: X → Y is a Multifunction, then the graph of F is the subset ∪ $\{\{x\} \times f(x): x \in X\}$ of X × Y. Graph of F is denoted by G(F).

Theorem: 4.1 Let m_X and m_Y be m-spaces and let $F: (X, m_X) \to (Y, m_Y)$ be multifunction. If the graph function G_F: X → $X \times Y$ is upper m_{wg}-continuous multifunction, then F is upper m_{wg}-continuous multifunction.

Proof: Suppose that G_F is upper m_{wg}-continuous. Let x ∈ X and W be any m-open set of m_Y such that F(x) ⊂ V. Then $G_F(x) \subset (X \times V)$ and $X \times V$ is m-open set in $X \times Y$. Since G_F is upper m_{wg}-continuous, there is an mwg-open set U with x ∈ U such that $G_F(U) \subset X \times V$. By Lemma 2.7 U ⊂ $G_F^+(X \times V)$ = X ∩ F^+(V) = F^+(V) and F(x) ⊂ V. So F is upper m_{wg}-continuous at x ∈ X.

Theorem: 4.2 A multifunction $F: (X, m_X) \to (Y, m_Y)$ lower m_{wg}-continuous multifunction if and only if the graph multifunction G_F is lower m_{wg}-continuous.

Proof: Suppose that F is lower m_{wg}-continuous multifunction. Let x ∈ X and W be any mwg-open set of $X \times Y$ such that x ∈ $G_F^-(W)$. Since W ∩ $\{\{x\} \times F(x)\}$ ≠ Φ, there exists y ∈ $F(x)$ such that (x, y) ∈ W and hence (x, y) ∈ $U \times V$ ⊆ W for some mwg-open sets of U and V of X and Y, respectively. Since $F(x) \cap V \neq \Phi$, there exists G ∈ m_X -WGO(X, x) such that G ⊆ F^-(V). By Lemma 2.7 U ∩ G ⊆ U ∩ F^-(V) = $G_F^-(U \times V)$ ⊆$G_F^-(W)$. Therefore, we obtain x ∈ U ∩ G ∈ m_X -WGO(X, x) and Hence G_F is lower m_{wg}-continuous.

Suppose that G_F is lower m_{wg}-continuous. Let x ∈ X and W be any m-open set of m_Y such that x ∈ F^-(V). Then $X \times V$ is mwg-open in $X \times Y$ and $G_F(x) \cap (X \times V)$ = ($\{x\} \times F(x)$) ∩ $(X \times V)$ = $\{x\} \times (F(x) \cap$ V) ≠ Φ. Since G_F is lower m_{wg}-continuous, there exists an mwg-open set U containing x such that U ⊆ $G_F^-(X \times V)$. By Lemma 2.7 we have U ⊆ F^-(V). This shows that F is lower m_{wg}-continuous multifunction.

Definition: 4.3 A nonempty set X with minimal structures m_X is said to be m_{wg}-compact if every cover of X by mwg-open sets has a finite subcover. A subset K of a nonempty set X with a minimal structure m_X is said to be m_{wg}-compact if every cover of K by mwg-open sets has a finite subcover.

Theorem: 4.4 Let (X, m_X) and (Y, m_Y) be nonempty sets with m- structures. If $F: (X, m_X) \to (Y, m_Y)$ is upper m_{wg}-continuous multifunction such that $F(x)$ is m_{wg}-compact for x ∈ X and K is a m_{wg}-compact set of (X, m_X), then $F(V)$ is m_{wg}-compact.

Proof: Let $\{V_i: i \in I\}$ be any mwg-open cover of $F(K)$. For each x ∈ K, F(x) is m_{wg}-compact and there exists a finite subset $I(x)$ of I such that $F(x) \subset \cup\{V_i: i \in I(x)\}$. Now, set $V(x) = \cup\{V_i: i \in I(x)\}$. Then we have $F(x) \subset V(x)$ and $V(x) \in m_Y$. Since F is upper m_{wg}-continuous, there exists $U(x) \in m_X$ containing x such that $F(U(x)) \subset V(x)$. The family $\{U(x) : x \in K\}$ is a cover of K by mwg-open sets. Since K is m_{wg}-compact, there exists a finite number of points, say, $x_1, x_2, .., x_n$, in K such that $K \subset \cup\{U(x_k) : x_k \in K, 1 \leq k \leq n\}$. Therefore, we obtain, $F(K) \subset \cup\{ F(U(x_k)) : x_k \in K, 1 \leq k \leq n\} \subset \cup\{V_i: i \in I(x_k) : x_k \in K, 1 \leq k \leq n\}$. This is shows that F(K) is m_{wg}-compact.

Theorem: 4.5 If $F: (X, m_X) \to (Y, m_Y)$ is an upper m_{wg}-continuous multifunction and $F(x)$ is m_{wg}-compact, then the graph multifunction G_F is upper m_{wg}-continuous.

Proof: Let x ∈ X and W be any mwg-open sets of $X \times Y$ containing$G_F(x)$. For each $y \in F(x)$, there exist mwg-open open sets $U(y) \subseteq X$ and $V(y) \subseteq Y$ such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family of $\{V(y) : y \in F(x)\}$ is an mwg-open cover of F(x). Since F(x) is m_{wg}-compact, it follows that there exists a finite number of points, says $y_1, y_2, \dots y_n$ in F(x) such that $F(x) \subseteq \{V(y_i) : i = 1, 2, \dots,n\}$. Take $U = \cap\{U(y_i) : i = 1, 2, \dots,n\}$ and $V = \cap\{V(y_i) : i = 1, 2, \dots,n\}$. Then U and V are mwg-open sets in X and Y, respectively, and $\{x\}\times F(x) \subseteq U \times V \subseteq W$. since F is an upper m_{wg}-continuous, there exist $U_0 \in m_X$ -WGO(X, x) such that $F(U_0) \subseteq V$. By Lemma 2.7, we have $U \cap U_0 \subseteq U \cap F^+(V) = G_F^+(U \times V) \in G_F^+(W)$. Therefore we obtain $U \cap U_0 \in m_X$ -WGO(X, x) and $G_F(U \cap U_0) \subseteq W$. This shows that G_F is upper m_{wg}-continuous.

Definition: 4.6 A space (X, m_X) is said to be m_{wg}- connected if it cannot be written as the union of two nonempty disjoint mwg-open sets.

Theorem: 4.7 Let $F: (X, m_X) \to (Y, m_Y)$ be a multifunction and X be a m_{wg}-connected space. If the graph multifunction G_F is upper m_{wg}-continuous (resp. lower m_{wg}-continuous), then F is upper m_{wg}-continuous (resp. lower m_{wg}-continuous).

Proof: Let x ∈ X and V be any open subset of Y containing $F(x)$. Since $X \times V$ is a mwg-open set of $X \times Y$ and $G_F(U) \subset X \times V$, there exist a mwg-open set U containing x such that $G_F(U) \subset X \times V$. By the Lemma 2.7, we have $U \subset G_F^+(X \times V) = F^+(V)$ and $F(U) \subseteq V$. Thus, F is upper m_{wg}-continuous. The proof of the lower m_{wg}-continuous F can be done using a similar argument.

Definition: 4.8 A graph of a multifunction $F: (X, m_X) \to (Y, m_Y)$ is said to be m_{wg}-closed if for each $(x, y) \in (X \times Y) - G(F)$, there exists $U \in m_X$ -WGO(X, x) and $V \in m_Y$ -WGO(Y, y) such that $(U \times V) \cap G(F) = \Phi$.

Lemma: 4.9 A multifunction $F: (X, m_X) \to (Y, m_Y)$ has a m_{wg}-closed if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exists $U \in m_X$ -WGO(X, x) and $V \in m_X$ -WGO(Y, y) such that $F(U) \cap V = \Phi$.
The proof is obivious.

Theorem: 4.10 If F: $(X, \ m_X) \rightarrow (Y, m_Y)$ is a point closed upper m_{wg}-continuous multifunction into a m-regular space, then F has a m_{wg}-closed graph.

Proof: Suppose (x, y) ∈ (X × Y) - G(F). Then y ∉ F(x). Thus there are disjoint m-open sets U, V ⊂ Y such that F(x) ⊂ U and y ∈ V. Since F is upper m_{wg}-continuous , there is an mwg-open set W ⊂ X containing x, such that F(W) ⊂ U. Thus (x, y) ∈ W × V and (W × V) ∩ G(F) = Φ. Hence, G(F) is m_{wg}-closed graph.

Theorem: 4.11 Let F: $(X, \ m_X) \rightarrow (Y, m_Y)$ be a multifunction from a space X into a m_{wg}-compact space Y. If G(F) is m_{wg}-closed, then F is upper m_{wg}-continuous.

Proof: Suppose that F is not upper m_{wg}-continuous. Then there exists a nonempty m-open subset C of Y such that F^{-}(C) is not mwg-open in X. We may assume F^{-}(C) ≠ Φ. Then there exists a point x_0 ∉ F^{-}(C). Hence for each point y ∈ C, we have (x_0, y) ∉ G(F). Since F has a m_{wg}-closed, there are mwg-open subsets U(y) and V(y) containing x_0 and y, respectively such that (U(y) × V(y)) ∩ G(F) = Φ. Then {Y\C} ∪ {V(y) : y ∈ C} is a mwg-open cover of Y, and Thus it has a subcover {Y\C} ∪ {V(y_i) : y_i ∈ C, 1 ≤ i ≤ n}. Let U = $\bigcap_{i=1}^{n} U(y_i)$ and V = $\bigcup_{i=1}^{n} V(y_i)$. It is easy to verify that C ⊂ V and (U × V) ∩ G(F) = Φ. Since U is mwg-neighbourhood of x_0, U ∩ F^{-}(C) ≠ Φ. It follows that Φ ≠ (U × C) ∩ G(F) ⊂ (U × V) ∩ G(F). This is contradiction. Hence proved.

Definition: 4.12 A graph of a multifunction F: $(X, \ m_X) \rightarrow (Y, m_Y)$ is said to be totally m_{wg}-closed if for each (x, y) ∈ (X × Y) - G(F), there exists U ∈ m_X -WGCO(X, x) and V ∈ m_Y -O(Y, y) such that (U × V) ∩ G(F) = Φ.

Lemma: 4.13 A multifunction F: $(X, \ m_X) \rightarrow (Y, m_Y)$ has a totally m_{wg}-closed if and only if for each (x, y) ∈ (X × Y) - G(F), there exists U ∈ m_X -WGCO(X, x) and V ∈ m_X -O(Y, y) such that $F(U) \cap V$ = Φ.
The proof is obivious.

Theorem: 4.14 Let F: $(X, \ m_X) \rightarrow (Y, m_Y)$ be a multifunction from a space X into a m-compact space Y. If G(F) is totally m_{wg}-closed, then F is upper m_{wg}-continuous.

Proof: Suppose that F is not upper m_{wg}-continuous. Then there exists a nonempty m-open subset C of Y such that F^{-}(C) is not mwg-open in X. We may assume F^{-}(C) ≠ Φ. Then there exists a point x_0 ∉ F^{-}(C). Hence for each point y ∈ C, we have (x_0, y) ∉ G(F). Since F has a totally m_{wg}-closed, there are mwg-open subsets U(y) and m-open subsets V(y) containing x_0 and y, respectively such that (U(y) × V(y)) ∩ G(F) = Φ. Then {Y\C} ∪ {V(y) : y ∈ C} is a m-open cover of Y, and Thus it has a subcover {Y\C} ∪ {V(y_i) : y_i ∈ C, 1 ≤ i ≤ n}. Let U = $\bigcap_{i=1}^{n} U(y_i)$ and V = $\bigcup_{i=1}^{n} V(y_i)$. It is easy to verify that C ⊂ V and (U × V) ∩ G(F) = Φ. Since U is mwg- neighbourhood of x_0, U ∩ F^{-}(C) ≠ Φ. It follows that Φ ≠ (U × C) ∩ G(F) ⊂ (U × V) ∩ G(F). This is contradiction. Hence the proof is completed.

Definition: 4.15 For a multifunction F: $(X, m_X) \rightarrow (Y, m_Y)$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be strongly m_{wg}-closed if for each (x, y) ∈ (X × Y) - G(F), there exists U ∈ m_X -WGO(X, x) and V ∈ m_Y -WGO(Y, y) such that (U × m_Y-Cl(V)) ∩ G(F) = Φ.

Lemma: 4.16 A multifunction F: $(X, m_X) \rightarrow (Y, m_Y)$ has a strongly m_{wg}-closed if and only if for each (x, y) ∈ (X × Y) - G(F), there exists U ∈ m_X -WGO(X, x) and V ∈ m_X -WGO(Y, y) such that $F(U) \cap m_Y - \mathrm{Cl}(V)$ = Φ.
The proof is obivious.

Theorem: 4.17 If F: $(X, m_X) \rightarrow (Y, m_Y)$ is upper m_{wg}-continuous multifunction such that F(x) is m_{wg}-compact for each x ∈ X and Y is a m-Urysohn space, then G(F) is strongly m_{wg}-closed.

Proof: Let (x, y) ∈ $(X \times Y) - G(F)$, then y ∈ $Y - F(x)$. Since Y is a m_{wg}-Urysohn space, there exist mwg-open sets V and W of Y such that y ∈ V, $F(x) \subset$ W and $m_Y - \mathrm{Cl}(V) \cap m_Y - \mathrm{Cl}(W)$ = Φ. Since F is upper m_{wg}-continuous, there exists U ∈ m_X -WGO(X, x) such that F(U) ⊂ $m_Y - \mathrm{Cl}(W)$. Therefore, we have $F(U) \cap m_Y - \mathrm{Cl}(V)$ = Φ. Hence G(F) is strongly m_{wg}-closed.

Theorem: 4.18 If F: $(X, m_X) \rightarrow (Y, m_Y)$ is an upper m_{wg}-continuous multifunction and $F(x)$ is m-compact. Also, Let $F(x) \cap F(y)$ = Φ for each pair of x, y ∈ X (x ≠ y). If Y is m-Haussdroff space, then X is m-Urysohn space.

Proof: Let $F(x) \cap G(x)$ = Φ for each pair of x, y ∈ X (x ≠ y). Since Y is m-Haussdroff space, $F(x)$ and $F(y)$ are m-compact sets, there exist m-open sets V, W ⊂ Y such that $F(x) \subset$ V, $F(y) \subset$ W such that V ∩ W = Φ. Since F is m_{wg}-continuous multifunction, there exist $U_1 \subset m_X - \mathrm{WGO}(X, x)$ and $U_2 \subset m_X - \mathrm{WGO}(X, y)$ such that x ∈ $m_X - \mathrm{Cl}(U_1) \subset F^+(V)$, y ∈ $m_X - \mathrm{Cl}(U_2) \subset F^+(W)$. Hence $m_X - \mathrm{Cl}(V) \cap m_X - \mathrm{Cl}(W)$ = Φ. Therefore, X is m-Urysohn space.

Theorem: 4.19 If F, G : $(X, m_X) \rightarrow (Y, m_Y)$ are upper m_{wg}-continuous multifunction into m-Urysohn space Y and for each x ∈ X, $F(x)$ and $G(x)$ are m-compact in (Y, m_Y), then U = { x ∈ X: $F(x) \cap G(x) \neq$ Φ} is wg-closed in (X, m_X).

Proof: Let x ∈ $X - A$. Then $F(x) \cap G(x) =$ Φ. Since Y is m-Urysohn space, there exist m-open sets P and Q such that $F(x) \subset P, G(x) \subset Q$ and $\mathrm{Cl}(P) \cap \mathrm{Cl}(Q) =$ Φ. Since F is upper m_{wg}-continuous, there exists $U_1 \subset m_X - \mathrm{WGO}(X, x)$ such that $F(U_1) \subset \mathrm{Cl}(P)$. Since G is upper m_{wg}-continuous, there exists $U_2 \subset m_X - \mathrm{WGO}(X, x)$ such that $F(U_2) \subset \mathrm{Cl}(Q)$. Now put $U = U_1 \cap U_2$, then we have $U \in m_X - \mathrm{WGO}(X, x)$ and $U \cap A =$ Φ. Therefore, A is mwg-closed in X.

Theorem: 4.20 F: $(X, m_X) \rightarrow (Y, m_Y)$ is an upper m_{wg}-continuous multifunction and m_{wg}-compact from a minimal space X to m-Urysohn space Y and let $F(x) \cap G(x) =$ Φ for each x, y (x ≠ y) ∈ X. Then X is m_{wg}-Haussdroff space.

Proof: Let x and y be any two distinct points in X. Then we have $F(x) \cap G(x) = \Phi$. Since Y is m-Urysohn space, there exist m-open sets P and Q such that $F(x) \subset U$, $G(x) \subset V$ and $m_Y - Cl(U) \cap m_Y - Cl(V) = \Phi$. Since F is upper m_{wg}-continuous, the $F(U)$ and $F(V)$ are disjoint mwg-open sets containing x and y respectively. Thus X is m_{wg}-Haussdroff space.

REFERENCES

1. J. Cao and I. L. Reilly, "α-Continuous and α-Irresolute Multifunction", *Mathematica Bohemica*, 121 (4), 415 – 424 (1996).
2. M. Mocanu, "On m-Compact spaces", Rendiconti Del Circolo Matematico Di Palermo, Series II, Tomo LIII, 1-26 (2005).
3. M. E. Abd El-Monsef and A. A. Nasef, "On Multifunction", Chaos, Solitons & Fractals, 12, (2001), 2387 - 2394.
4. N. Nagaveni and P. Sundaram "On weakly generalized continuous maps, weakly generalized closed maps and weakly generalized irresolute maps in topological spaces", *Far East Journal of Math. Sci.* 6 (6), 903 – 912 (1998).
5. T. Noiri and V. Popa, "Almost Weakly Continuous Multifunction", *Demonstratio Math.* 26, 363 – 380 (1993).
6. T. Noiri and V. Popa, "On Upper and Lower M -Continuous Multifunction", *Filomat*, 14, 73 – 86 (2000).
7. T. Noiri and V. Popa, "Slightly m-Continuous Multifunction", Bulletin of the Institute of mathematics, *Academia Sinica (new series)* 1(4), 485 – 505 (2006).
8. T. Noiri and V. Popa, "A Unified Theory of Upper and Lower Almost Nearly Continuous Multifunction", *Mathematica Balkanica*, 23, 51 – 72 (2009).
9. R. Parimelazhagan, K. Balachandran and N. Nagaveni, "Weakly Generalized closed sets in Minimal Structures", *Int. J. Contemp. Math. Sciences*, 4 (27), 1335 -1343 (2009).
10. R. Parimelazhagan, N. Nagaveni and Sai Sundara Krishnan, "On mg-Continuous Functions in Minimal Structure", *Proc. Int. Conf. Engineers and Computer Scientists* vol I, 18 – 20 (2009).
11. V. Popa, "On Almost Continuous Multifunction", Review of research, *Faculty of Science University of Novi Sad Mathematics Series*, 15 (1), 145 – 154 (1985).
12. D. Sheeba and N. Nagaveni, "On Minimal Topological Totally Closed graphs", *Annals of Pure and Applied Mathematics*, 16 (2), 401 – 411 (2018).
13. R. E. Smithson, "Subcontinuity for Multifunction", *Pacific Journal of Mathematics*, 61 (1), 283 – 288 (1975).