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Fixed Points of Asymptotically Regular Mappings in Metric Space

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ABSTRACT

Determining the fixed point theorem by using the intension of work Slobodan[12], we have proved the fixed point theorems in metric spaces with the help of condition [2.5]

Keywords: Asymptotically Regular mapping, Metric space, Complete metric space, Cauchy sequence.

1. INTRODUCTION

Browder and Petryshyn[1] introduced the concept of asymptotically regular mapping at a point in Banach space. In 1969 Nadler[9] introduced the concept of multi valued contraction mappings and established that a multi valued contraction mapping possesses a fixed point in a complete metric space. Jungok[7] generalized the Banach contraction principle by using a contraction condition for a pair of commuting mappings in a metric space. Jungok pointed out the potential of commuting mappings for generalizing fixed point theorems in[7,8].

2. PRELIMINARIES AND DEFINITIONS

2.1 Asymptotically regular mapping

Let (X,d) be a metric space and T is a self mapping of X. Then T is said to be asymptotically regular mapping if $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$.

2.2 Let A and S be two self maps of X and seq. $\{x_n\}$ in X. Then $\{x_n\}$ is said to be asymptotically S regular with respect to A if $d(Ax_n, Sx_n)\to 0$ as $n\to\infty$.

2.3 A space X is said to be T-orbitally completes iff every Cauchy sequence of the form $\{(x_{n_i}, x_{n_i} \in T(x_{n_{i-1}})\}$ converges in X.

2.4 T and f are said to commute if for each $x\in X$, $f(T(x))=fTx\subseteq Tfx = T(f(x))$.

In this paper, we use the condition.

2.5 $$d(Tx, Ty) \le q \max \Big\{d(x,y), \frac{d(x, Ty)\,[1 + d(x, Tx)]}{[1 + d(x, y)]}, \frac{1}{2}\frac{d(y, Tx)\,[1+d(x, Tx) + d(y, Ty)]}{[1+d(x, y)]}, \frac{1}{2}\frac{d(x, Tx)\,[1+d(x, Tx) + d(y, Tx)]}{[1+d(x, y)]},$$

where x, y ∈ X and X is a metric space.

We have extended the work of Slobodan[12] for asymptotically regular mappings satisfying the condition [2.5].

3. MAIN RESULTS

3.1 Theorem : Let (X,d) be a complete metric space and T:X→X be a mapping such that the following condition is satisfied

$$d(Tx, Ty) \le q \max\left\{\frac{d(x, Ty)\,[1+d(x,Tx)]}{[1+d(x,y)]}, \frac{1}{2}\frac{d(y, Tx)\,[1 + d(x, Tx) + d(y, Ty)]}{[1 + d(x, y)]}, \frac{1}{2}\frac{d(x,Tx)\,[1 + d(x,Ty) + d(y,Tx)]}{[1 + d(x,y)]}\right\}$$

∀ x, y ∈ X, where 0 < q < 1. If T is asymptotically regular at some point of X, then T has a unique fixed point in X.

Proof: Let T be asymptotically regular at x_0 ∈ X. Consider the sequence $\{T^n x_0\}$.
From condition [2.5] and the triangular inequality, we have

$$d(T^m x_0, T^n x_0) \le q \max \Big\{d(T^{m-1} x_0, T^{n-1} x_0), \frac{d(T^{m-1}x_0, T^n x_0)\,[1+ d(T^{m-1}x_0, T^m x_0)]}{[1+d(T^{m-1}x_0, T^{n-1}x_0)]},$$

$$\frac{1}{2}\frac{d(T^{n-1}x_0, T^m x_0)\,[1+d(T^{m-1}x_0, T^m x_0) + d(T^{n-1}x_0, T^n x_0)]}{[1+d(T^{m-1}x_0, T^{n-1}x_0)]},$$

$$\frac{1}{2}\frac{d(T^{m-1}x_0, T^n x_0)\,[1+d(T^{m-1}x_0, T^n x_0) + d(T^{n-1}x_0, T^m x_0)]}{[1+d(T^{m-1}x_0, T^{n-1}x_0)]}\Big\}$$

$$\le q \max \{d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0),$$

$$\frac{d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0)].[1+ d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0)]}{[1+d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0)]},$$

$$\frac{1}{2}\frac{[d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0)].[1+ d(T^{m-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0) + d(T^{n-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0)]}{[1+d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0)]},$$

$$\frac{1}{2}\frac{[d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0)].[1+ d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0)]}{[1+d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0)]}\},$$

$$\Rightarrow d(T^m x_0, T^n x_0) \le q[d(T^{n-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0)],$$

$$\Rightarrow (1-q)d(T^m x_0, T^n x_0) \le q[d(T^{n-1}x_0, T^m x_0) + d(T^n x_0, T^{n-1}x_0)]$$

$$\Rightarrow d(T^m x_0, T^n x_0) \le \left(\frac{q}{1-q}\right)[d(T^{n-1}x_0, T^m x_0) + d(T^n x_0, T^{n-1}x_0)]$$

$\therefore$ T is asymptotically regular at x_0, we have $d(T^m x_0, T^n x_0) \to 0$ as n, $m \to \infty$
Hence $\{T^n(x_0)\}$ is a Cauchy sequence.
Since (X, d) is a complete metric space,
$\exists$ a point u in X, such that
$u = \lim_{n\to\infty} T^n x_0$.
Suppose u is not a fixed point of T.
By condition [2.5] we obtain
$d(u, Tu) \le d(u, T^n x_0) + d(T^n x_0, Tu)$
$\le d(u, T^n x_0) + q. \max \{d(T^{n-1}x_0, u),$

$$\frac{d(T^{n-1}x_0, Tu)[1 + d(T^{n-1}x_0, T^n x_0)]}{1 + d(T^{n-1}x_0, u)},$$

$$\frac{1}{2}\frac{d(u, T^n x_0)[1 + d(T^{n-1}x_0, T^n x_0) + d(u, Tu)]}{1 + d(T^{n-1}x_0, u)},$$

$$\frac{1}{2}\frac{d(T^{n-1}x_0, T^n x_0)[1 + d(T^{n-1}x_0, Tu) + d(u, T^n x_0)]}{1 + d(T^{n-1}x_0, u)},$$

Taking limit as $n\to\infty$
$d(u, Tu) \le q \max \{0, d(u, Tu), 0, 0\}$
$= q\, d(u, Tu)$
$\Rightarrow d(u, Tu) \le q\, d(u, Tu)$
$\Rightarrow (1-q)\, d(u, Tu) \le 0$
which is a contradiction $q < 1$ unless $u = Tu$
Suppose that T has a second fixed point V in X. Then $d(u,v) \le q\, d(u,v)$
Since $q < 1$ it follows that $u = v$
$\therefore$ Fixed point is unique
Hence the proof

3.2 Theorem

Let (X,d) be a metric space and T be a mapping of X into itself satisfying the condition (2.5) of theorem (3.1), where $0 < q < 1$. If T is asymptotically regular at a point x in X and the sequence of iterates $\{T^n x_0\}$ has a subsequence converging to a point z in X, then z is the unique fixed point of T and $\{T^n x_0\}$ also converging to z.

Proof: Let T be asymptotically regular at x in X consider the sequence $\{T^n x\}$. suppose that $\lim_K T^{n_k} x = z$ and $Tz \ne z$.
By condition, we have
$d(z, Tz) \le d(z, T^{n_k} x) + d(T^{n_k} x, T^{n_k+1} x) + d(T^{n_k+1} x, Tz) \le d(z, T^{n_k} x) + d(T^{n_k} x, T^{n_k+1} x) + q \max \{d(T^{n_k} x, z),$

$$\frac{d(T^{n_k}x, Tz)[T + d(T^{n_k}x, T^{n_k+1}x)]}{1 + d(T^{n_k}x, z)},$$

$$\frac{1}{2}\frac{d(z, T^{n_k+1}x)[1 + d(T^{n_k}x, T^{n_k+1}x) + d(z, Tz)}{1 + d(T^{n_k}x, z)}$$

$$\left.\frac{1}{2}\frac{d(T^{n_k}x, T^{n_k+1})[1 + d(T^{n_k}x, Tz) + d(z, Tz)}{1 + d(T^{n_k}x, z)}\right\}$$

$\le d(z, T^{n_k} x) + d(T^{n_k} x, T^{n_k+1} x) + q \max \{d(T^{n_k} x, z),$

$$\frac{[d(T^{n_k}x, z) + d(z, Tz)][1 + d(T^{n_k}x, T^{n_k+1}x)]}{1 + d(T^{n_k}x, z)},$$

$$\frac{1}{2}\frac{d(z, T^{n_k+1}x)[1 + d(T^{n_k}x, T^{n_k+1}x) + d(z, Tz)]}{1 + d(T^{n_k}x, z)},$$

$$\left.\frac{1}{2}\frac{d(T^{n_k}x, T^{n_k+1}x)[1 + d(T^{n_k}x, Tz) + d(z, T^{n_k+1}x)]}{[1 + d(T^{n_k}x, z)]}\right\}$$

Taking the limit as $k\to\infty$, we obtain

$d(z, Tz) \leq q \max \{0, d(z, Tz), 0\}$

$\Rightarrow\ d(z, Tz) \leq qd(z, Tz)$

which contradiction $q < 1$ unless $z = Tz$.

By theorem , z is the unique fixed point of T.

Now we can prove that $\lim_{n\to\infty} T^{n_k} x = z$

We have

$$\begin{aligned} d(z, T^n x) &= d(Tz, T^n x) \\ &\leq d(Tz, T^{n+1}x) + d(T^{n+1}x, T^n x) \\ &\leq q \max \Bigg\{ d(z, T^n x), \frac{d(z,T^n x)[1+d(z,Tz)]}{[1+d(z,T^n x)]}, \\ &\quad \frac{1}{2}\frac{d(T^n x, Tz)[1+d(z,Tz)+d(T^n x, T^{n+1}x)]}{[1+d(z,T^n x)]}, \\ &\quad \frac{1}{2}\frac{d(z,Tz)[1+d(z,T^{n+1}x)+d(T^n x, Tx)]}{1+d(z,T^n x)} \Bigg\} \\ &\quad + d(T^{n+1}x, T^n x) \end{aligned}$$

$\Rightarrow\quad d(z,T^n x) \leq qd(z, T^n x) + d(T^{n+1}x, T^n x)$

$\Rightarrow\quad (1-q)\, d(z,T^n x) \leq d(T^{n+1}x, T^n x)$

$\therefore\quad q < 1$ and $Tz = z$

Since T is asymptotically regular at x.

Hence $d(T^{n+1}x, T^n x) = 0$ as $n\to\infty$.

$\therefore\quad \{T^n(x)\}$ converges to z.

This completes the theorem.

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