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# Best Proximity Point Theorems of C-class Functions in Regular Cone Metric Spaces

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**ABSTRACT**

The purpose of this article is using the concept of cone C-class functions, we prove the existence of best proximity point on A × B, where A and B are nonempty subsets of regular cone metric spaces.

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## 1. INTRODUCTION AND PRELIMINARIES

Let A and B are two nonempty subsets of a metric space X. If there is a pair $d(x_0, y_0) = d(A, B)$, that $d(A, B) = \inf\{d(s, t) : s \in A, t \in B\}$, then the pair $(x_0, y_0)$ is called the best proximity point for A and B. We can find the best proximity pair of the sets A and B, by considering a map $T: A\cup B \to A\cup B$ such that
$T(A) \subset B$, $T(B) \subset A$. The point $x \in A \cup B$ is a best proximity point for T if,
$d(x, T(x)) = d(A, B)$.

A map $T: A\cup B \to A\cup B$, such that $T(A) \subset B$, $T(B) \subset A$ is called cyclic contraction[3] if, for some $k \in [0, 1)$ the condition $d(T(x), T(y)) \leq kd(x, y) + (1 - k)d(A, B)$, holds for all $x \in A$, $y \in B$.

In 2003, Kirk *et al.* proved fixed point results for cyclic contraction maps[8]. In 2006, Eldred and Veeramani obtained best proximity point results for cyclic contraction maps[3].

Best proximity point theory of cyclic contraction maps has been studied by many authors see[1,3,9] and references therein. In 2007, Huang and Zhang[6] introduced cone metric spaces as a generalisation of metric spaces. Then in[10] some results about characterisation best approximations in the cone metric spaces are studied. In 2011, Haghi *et al*.[4] obtained best proximity point for cyclic contraction maps in cone metric spaces. In 2012, Karapinar[7], obtained best proximity points for certain cyclic contraction maps in metric spaces. In 2013, Amini *et al*.[2], introduce a new class of cyclic generalised contraction maps and it is shown that the best proximity point property for closed convex subsets of a uniformly convex banach space holds.

In this paper, we prove existence of best proximity point theorems for various types of cyclic contraction maps, which are generalisation of some results in the literature. To prove our results in the next section we recall some definitions and facts. In the sequel E stands for a real Banach space.

A subset P of E is called a cone if it satisfies the following:

(i). P is closed, nonempty and $P \neq \{0\}$.

(ii). $a, b \in R^+$ and $x, y \in P$ implies $ax + by \in P$ .

(iii). $x \in P$ and $-x \in P$ implies $x = 0$.

We define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$ .

$x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where intP denotes the interior of P .

A map $f : P \to P$ is said to be increasing (strictly increasing) whenever $x \leq y$ implies that $f(x) \leq f(y)$ ($x < y$ implies that $f(x) < f(y)$ ).

A cone is said to be normal if there is a number $M > 0$ such that for all $x, y \in E$

$0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.

The least positive number M satisfying the above inequality is called the normal constant of cone P .[6]

**Theorem 1.1.**[3] Let A and B are two nonempty subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Let $x_0 \in A$ and define $T(x_n)=x_{n+1}$ . Suppose $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, T(x)) = d(A, B)$.

**Definition 1.2.** Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies for all $x, y, z \in X$

($d_1$) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

($d_2$) $d(x, y) = d(y, x)$

($d_3$) $d(x, y) \leq d(x, z) + d(z, y)$

Then d is called a cone metric and (X, d) is called a cone metric space.

**Definition 1.3.**[12] A non-empty subset A of (X, d), is said to be bounded above if there exists $c \in intP$ such that $c - d(x, y) \in P$ for all $x, y \in A$.

The cone is called regular if every sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq y$ for some $y \in E$, then there ia $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently,, the cone P is regular if and only if, for every decreasing sequence which is bounded from below is convergent.

**Lemma 1.4.**[11] Every regular cone is normal.

**Example 1.5.**[5] Let $E = (L^1[0, 1], \| .\|_1)$, $P = \{f \in E : f \geq 0 \text{ a.e.}\}$, $(X, \rho)$ be a metric space and $d : X \times X \to E$ defined by $d(x, y) = f_{x,y}$, where $f_{x,y}(t) = \rho(x, y)t^2$. Then $(X, d)$ is a regular cone metric space. In fact, if $\{f_n\}_{n\geq 1}$ is an increasing sequence and there is $g \in L^1$ such that $f_1 \leq f_2 \leq ... \leq f_n \leq ... \leq g$ for almost everywhere x, then $\{f_n\}_{n\geq 1}$ converges to a function f a.e on X. Then, $f_n \leq f \leq g$ (a.e) for all $n \geq 1$. Thus $g - f_1 \in L_1$, $g - f_n \leq g - f_1$ for all $n \geq 1$ and $\lim_{n\to\infty} g - f_n = g - f$ (a.e). Hence by the lebesgue dominated convergence theorem, $f \in L^1$ and $\lim_{n\to\infty} \|f_n - f_1\|_1 = 0$. So, the cone P is regular.

**Definition 1.6.**[4] Let A and B are nonempty subsets of cone metric space (X, d).
An element $p \in P$ is said to be a lower bound for $A \times B$ whenever

$$p \leq d(a, b), \tag{1.1}$$

for all $(a, b) \in A \times B$. If $p \geq q$ for all lower bound q for $A \times B$, then p is called greatest lower bound for $A \times B$. We denote it by $d(A, B)$. Clearly, $d(A, B)$ is unique vector in P .

**Definition 1.7.**[4] A map $\Psi : P \to P$ is called cone $L_1$ −function whenever $\Psi(0) = 0$ ,$\Psi(s) > 0$ for all $s \in P$ with $s\neq 0$ and there exists $\delta_s \gg 0$ such that $\Psi(t) < s$ for all $s \leq t \leq s + \delta_s$.

**Definition 1.8.** A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:
(i) $\psi$ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.
Also the ultra altering distance functions were introduced by H. Ansari. in[13] and now we define this functions on a cone. If $P := R +$ then we have the definition

**Definition 1.9.** An ultra altering distance function is a function $\varphi : P \to P$ which satisfies
(a) $\varphi$ is continuous.
(b) $\varphi(0) \geq 0$.

**Lemma 1.10.** Let $\psi$ and $\varphi$ are altering distance and ultra altering distance functions respectively, $F \in C$ and $\{s_n\}$ a decreasing sequence in P such that
$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$
for all $n \geq 1$. Then $\lim_{n\to\infty} s_n = 0$.

**Definition 1.11.**[14] A mapping $F : P^2 \to P$ is called cone C-class function if it is continuous and satisfies following axioms:
(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in P$ .
We denote cone C-class functions as C.

**Example 1.12.**[14] The following functions $F:[0,\infty)^2\to R$ are elements of C,

for all s, t ∈ [0, ∞):
(1)F (s, t) = s − t,
(2)F (s, t) = ks, where 0<k<1,
(3)F (s, t) = sβ(s), where β : [0, ∞) → [0, 1),
(4)F (s, t) = Ψ(s), where Ψ : P → P is a cone $L_1$ −function,
(5)F (s, t) = s − φ(s), where φ : [0,∞) → [0,∞) is a continuous function such that φ(t)=0 ⇔ t = 0;
(6) F (s, t) = s − h(s, t), where h : [0, ∞) × [0, ∞) → [0, ∞) is a continuous function such that h(s, t) = 0 ⇔ t = 0 for all t, s > 0.

## 2. MAIN RESULTS

Throughout this section, E is a norned linear space, (X, d) is a regular cone metric space, ≤ is the partial ordering with respect to P and A, B are nonempty subsets of X.

**Theorem 2.1.** Let T : A ∪ B → A ∪ B be a map such that T (A) ⊂ B, T (B) ⊂ A and
d(T (x), T (y) ≤ F(max {d(x, T (y)), ½[d(x, T (x)) + d(y, T (y))]} − d(a, b),
φ (max {d(x, T (y)), ½[d(x, T (x)) + d(y, T (y))]}− d(a, b))) + d(a, b)
for all (a, b), (x, y) ∈ A × B, where φ is ultra altering distance function and F ∈ C such that s ≤ F (s, φ(s)) implies that s ≤ 0. Then, d(A, B) exists.

Proof. Let $x_0$ ∈ A ∩ B, set $x_{n+1}$ = T ($x_n$) and $d_{n+1}$ = d($x_{n+1}$, $x_n$) for all n ≥ 1. Then,

$$
\begin{aligned}
d_{n+1} &= d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1}))\\
&\leq F(\max\{d(x_n, T(x_{n-1})), \tfrac{1}{2}[d(x_n, T(x_n)) + d(x_{n-1}, T(x_{n-1}))]\} - d(a, b),\\
&\quad \varphi(\max\{d(x_n, T(x_{n-1})), \tfrac{1}{2}[d(x_n, T(x_n)) + d(x_{n-1}, T(x_{n-1}))]\} - d(a, b)))\\
&\quad + d(a, b)\\
&\leq F(\max\{d(x_n, x_n), \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} - d(a, b),\\
&\quad \varphi(\max\{d(x_n, x_n), \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} - d(a, b))) + d(a, b)\\
&= F(\tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] - d(a, b), \varphi(\tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] -\\
&\qquad d(a, b))) + d(a, b) \qquad (2.1)\\
&= F(\tfrac{1}{2}[d_{n+1} + d_n] - d(a, b), \varphi(\tfrac{1}{2}[d_{n+1} + d_n] - d(a, b))) + d(a, b)\\
&\leq \tfrac{1}{2}[d_{n+1} + d_n]
\end{aligned}
$$

It follows that $d_{n+1} \leq d_n$ for all n ≥ 1. By the regularity of the cone P , there exists p ∈ P such that $d_n \to p$ as n → ∞.
Thus from we have
p ≤ F((p − d(a, b)), φ (p − d(a, b))) + d(a, b)
so, p ≤ d(a, b) holds for all (a, b) in (A × B). Now if q is a lower bound for d(a, b) for all (a, b) in A × B, then q ≤ $d_n$ for all n ≥ 1, and so, q ≤ p.
Therefore, d(A, B) = p.

**Theorem 2.2.** Suppose that the conditions of Theorem (2.1) hold, $x_0$ ∈ A and $x_{n+1}$ = T ($x_n$) for all n ≥ 1. If {$x_{2n}$} has a convergent subsequence in A, then there exists x ∈ A such that d(x, T (x)) = d(A, B).

Proof. Let {$x_{2nk}$} be the convergent subsequence of {$x_{2n}$} in A with {$x_{2nk}$}→x ∈ A.

Since
$p = d(A, B) \leq d(x, x_{2nk-1}) \leq d(x, x_{2nk}) + d(x_{2nk}, x_{2nk-1})$
for each $k \geq 1$ and $d(x_{2nk}, x_{2nk-1})$ is a subsequence of $d_n$, from above inequality we get $\lim_{n\to\infty} (x, x_{2nk-1}) \to p$. As,

$$p \leq d(x_{2nk}, T(x_{2nk})) \leq d(x_{2nk-1}, x_{2nk})$$

for each $k \geq 1$. It follows that $d(x, T(x)) = p = d(A, B)$.

**Theorem 2.3.** Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x), T(y)) \leq F(\max\{d(x, y), ½[d(x, T(x)) + d(y, T(y))]\} - d(a, b),$$
$$\varphi(\max\{d(x, T(y)), ½[d(x, T(x)) + d(y, T(y))]\} - d(a, b))) + d(a, b)$$

for all $(a, b), (x, y) \in A \times B$, where $\varphi$ is ultra altering distance function and $F \in C$ such that $s \leq F(s, \varphi(s))$ implies that $s \leq 0$. Then, $d(A, B)$ exists.

Proof. Let $x_0 \in A \cap B$, set $x_{n+1} = T(x_n)$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. Then,

$$\begin{aligned} d_{n+1} &= d(x_{n+1}, x_n) \\ &= d(T(x_n), T(x_{n-1})) \\ &\leq F(\max\{d(x_n, x_{n-1}), ½[d(x_n, T(x_n)) + d(x_{n-1}, T(x_{n-1}))]\} - d(a, b), \\ &\quad \varphi(\max\{d(x_n, x_{n-1}), ½[d(x_n, T(x_n)) + d(x_{n-1}, T(x_{n-1}))]\} - d(a, b))) + d(a, b) \\ &\leq F(\max\{d(x_n, x_{n-1}), ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} - d(a, b), \\ &\quad \varphi(\max\{d(x_n, x_{n-1}), ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} - d(a, b))) + d(a, b) \quad (2.2) \\ &\leq \max\{d(x_n, x_{n-1}), ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} \end{aligned}$$

if $\max\{d(x_n, x_{n-1}), ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} = ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$ ,then
$d_{n+1} \leq ½ [d_{n+1} + d_n]$
which is equivalent to
$d_{n+1} \leq d_n$
if $\max\{d(x_n, x_{n-1}), ½[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]\} = d(x_n, x_{n-1})$ ,then
$d_{n+1} \leq d_n$
Therefore, it follows that $d_{n+1} \leq d_n$ for all $n \geq 1$.By the regularity of the cone P , there exists $p \in P$ such that $d_n \to p$ as $n \to \infty$. Thus from (2.2) we have
$p \leq F(p - d(a, b), \varphi(p - d(a, b))) + d(a, b)$
so, $p \leq d(a, b)$ holds for all $(a, b)$ in $(A \times B)$. Now if q is a lower bound for $d(a, b)$ for all $(a, b)$ in $A \times B$, then $q \leq d_n$ for all $n \geq 1$, and so, $q \leq p$. Therefore, $d(A, B) = p$.
With choice $F(s, t) = ks$, where $0<k<1$ in Theorem (2.3) we have the following corollary.
The above equality satisfies the condition $s \leq F(s, \varphi(s))$ implies that $s \leq 0$.

**Corollary 2.4.**[16] Let $T : A\cup B \to A\cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x), T(y)) \leq k \max\{d(x, y), ½[d(x, T(x)) + d(y, T(y))]\} + (1 - k)d(a, b) \quad (2.3)$$

for all $(a, b), (x, y) \in A \times B$, for some $k \in (0, 1)$. Then, $d(A, B)$ exists.

**Theorem 2.5.** Let $T : A\cup B \to A\cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x),T(y)) - d(a, b) \leq F((1/a+b+c)[ad(x, y) + bd(x, T(x)) + cd(y, T(y))] - d(a, b),$$
$$\varphi((1/a+b+c)[ad(x, y) + bd(x, T(x)) + cd(y, T(y))] - d(a, b)))$$

for all (a, b), (x, y) $\in A \times B$, where $\varphi$ is ultra altering distance function and $F \in C$ such that $s \le F(s, \varphi(s))$ implies that $s \le 0$ .and a, b, c are constants such that a, b, $c \ge 0$ with $a + b + c > 0$. Then, d(A, B) exists.

Proof. Take $x_0 \in A \cap B$. Set $x_{n+1} = T(x_n)$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \ge 1$. Then

$$\begin{aligned} d_{n+1} - d(a, b) &= d(x_{n+1}, x_n) - d(a, b) \\ &= d(T(x_n), T(x_{n-1})) - d(a, b) \\ &\le F((1/a+b+c)[ad(x_n,x_{n-1}) + bd(x_n,T(x_n)) + cd(x_{n-1},T(x_{n-1}))] - d(a, b), \\ &\quad \varphi((1/a+b+c)[ad(x_n,x_{n-1})+bd(x_n,T(x_n))+cd(x_{n-1},T(x_{n-1}))] - d(a, b))) \\ &= F((1/a+b+c)[ad(x_n, x_{n-1}) + bd(x_n, x_{n+1}) + cd(x_{n-1}, x_n)] - d(a, b), \\ &\quad \varphi((1/a+b+c)[ad(x_n, x_{n-1}) + bd(x_n, x_{n+1}) + cd(x_{n-1}, x_n)] - d(a, b))) \\ &= F((1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b), \\ &\quad \varphi((1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b))) \\ &\le (1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b) \end{aligned}$$

which implies

$$\begin{aligned} d_{n+1} - d(a, b) \le F((1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b), \\ \varphi((1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b))) \end{aligned} \tag{2.4}$$

and

$$d_{n+1} - d(a, b) \le (1/a+b+c)[ad_n + bd_{n+1} + cd_n] - d(a, b)$$

for all (a, b) $\in A \times B$, which is equivalent to

$$d_{n+1} \le d_n$$

Hence, $d_{n+1} \le d_n$ for all $n \ge 1$. By the regularity of the cone P , there exists $p \in P$ such that $d_n \to p$ as $n \to \infty$. Thus from (2.4) we have

$$p \le F(p - d(a, b), \varphi(p - d(a, b))) + d(a, b)$$

so, $p \le d(a, b)$ holds for all (a, b) in $(A \times B)$.

Now if q is a lower bound for d(a, b) for all (a, b) in $A \times B$, then $q \le d_n$ for all $n \ge 1$, and so, $q \le p$. Therefore, d(A, B) = p.

With the choice that F (s, t) = (a + b + c)s, where 0< (a + b + c) <1 in Theorem (2.5) we have the following corollary. The above equality satisfies the condition $s \le F(s, \varphi(s))$ implies that $s \le 0$.

**Corollary 2.6.** Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x), T(y)) \le ad(x, y) + bd(x, T(x)) + cd(y, T(y)) + ed(a, b)$$

for all (a, b), (x, y) $\in A \times B$, where a, b, c are constants such that $a + b + c < 1$ and $a + b + c + e \le 1$. Then, d(A, B) exists.

Proof. Suppose $a + b + c + e \le 1$ implies $e \le 1-a-b-c$ and hence,

$$\begin{aligned} d(T(x), T(y)) &\le ad(x, y) + bd(x, T(x)) + cd(y, T(y)) + (1-a-b-c)d(a, b) \\ &\le F((1/a+b+c)[ad(x, y) + bd(x, T(x)) + cd(y, T(y))] - d(a, b), \\ &\quad \varphi((1/a+b+c)ad(x, y)+bd(x, T(x))+cd(y,T(y))]-d(a, b))) + d(a, b) \end{aligned}$$

Then, by Theorem (2.5) we have d(A, B) = p.

Next we give the theorem for the case that b = c in the above corollary.

**Corollary 2.7.** Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x), T(y)) \leq ad(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + ed(a, b)$$

for all $(a, b), (x, y) \in A \times B$, where a, b, c are constants such that $a + 2b < 1$ and $a + 2b + e = 1$. Then, $d(A, B)$ exists.

**Theorem 2.8.**[16] Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$ and

$$d(T(x), T(y)) \leq ad(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + cd(a, b)$$

for all $(a, b), (x, y) \in A \times B$, where a, b, c are constants such that $a + 2b + c < 1$. Then, $d(A, B)$ exists.

Proof.

$$\begin{aligned} d(T(x), T(y)) &\leq ad(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + cd(a, b) \\ &\leq ad(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + (1-a-2b)d(a, b) \end{aligned}$$

Therefore from Corollary (2.7), $d(A, B)$ exists.

**Theorem 2.9.** Suppose that the conditions of Theorem (2.5) hold, $x_0 \in A$ and $x_n \in T(x_n)$ for all $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, T(x)) = d(A, B)$.

Proof. Proof is similiar to the proof of Theorem (2.2).

**Theorem 2.10.** Let $S, T : A \cup B \to A \cup B$ be a map such that $S(A) \subset B$, $T(B) \subset A$ and

$$d(S(x), T(y)) - d(a, b) \leq F(d(x, y) - d(a, b), \varphi d(x, y) - d(a, b))) \qquad (2.5)$$

for all $(a, b), (x, y) \in A \times B$, where $\varphi$ is ultra altering distance function and $F \in C$ such that $s \leq F(s, \varphi(s))$ implies that $s \leq 0$. Then, $d(A, B)$ exists.

Proof. Take $x_0 \in A \cap B$, then $S(x_0) \in B$, so there exists $y_0 \in B$ such that $y_0 = S(x_0)$. Set $T(y_0) \in A$, so there exists $x_1 \in A$ such that $x_1 = T(y_0)$. Inductively, define the sequence $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = T(y_n), \quad y_n = S(x_n) \qquad (2.6)$$

Set $d_n = d(x_n, s(x_n))$. Since

$$\begin{aligned} d_{n+1} - d(a, b) &= d(x_{n+1}, S(x_{n+1})-d(a, b) \\ &= d(T(y_n), S(x_{n+1})) - d(a, b) \\ &\leq F(d(y_n, x_{n+1}) - d(a, b), \varphi(d(y_n, x_{n+1})-d(a, b))) \\ &\leq d(y_n, x_{n+1}) - d(a, b) = d(S(x_n), T(y_n) - d(a, b)) \\ &\leq F(d(y_n, x_n) - d(a, b), \varphi(d(y_n, x_n), -d(a, b))) \\ &= F(d(S(x_n), x_n) - d(a, b), \varphi(d(S(x_n), x_n) - d(a, b))) \\ &= F(d_n - d(a, b), \varphi(d_n - d(a, b))) \end{aligned}$$

implies that

$$d_{n+1} - d(a, b) \leq F(d_n - d(a, b), \varphi(d_n - d(a, b))) \leq d_n - d(a, b)$$

for all $(a, b) \in A \times B$. Hence, $d_{n+1} \leq d_n$ for all $n \geq 1$.

Similiar to the proof of Theorem(2.1) we obtain the result.

Also, in case $S = T$ , Theorem (2.10) reduce to the following Theorem.

**Corollary 2.11.** Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subset B$, $T(B) \subset A$

and

$d(T(x), T(y)) - d(a, b) \leq F(d(x, y) - d(a, b), \varphi d(x, y) - d(a, b)))$ (2.7)

for all $(a, b), (x, y) \in A \times B$, where $\varphi$ is ultra altering distance function and $F \in C$ such that $s \leq F(s, \varphi(s))$ implies that $s \leq 0$. Then, $d(A, B)$ exists.

**Theorem 2.12.** Suppose that the conditions of Theorem (2.10) hold and the sequence $\{x_n\}$ and $\{y_n\}$ are generated by (2.6) for some $x_0 \in A$. If both $\{x_n\}$ and $\{y_n\}$ have a convergent subsequence in A and B respectively, then there exists $x \in A$ and $y \in B$ such that $d(x, S(x)) = d(A, B) = d(y, T(y))$.

Proof. Let, $d_n = d(x_n, S(x_n))$. Let $\{y_{nk}\}$ be the subsequence of $\{y_n\}$ such that $\{y_{nk}\} \to y$.

The relation $d(A, B) = p \leq d(y, T(y_{nk}) \leq d(y, y_{nk}) + d(y_{nk}, T(y_{nk}))$ (2.8)

holds for each $k \geq 1$. Since

$d(y_{nk}, T(y_{nk}) - d(a, b) \leq F(d_{nk} - d(a, b), \varphi(d_{nk} - d(a, b)))$

for all $(a, b) \in A \times B$, it follows that $d(y_{nk}, T(y_{nk})) \leq d_{nk}$. Since $\{d(x_{nk}, S(x_{nk}))\}$ is a subsequence of $\{d_n\}$, hence $\lim_{n\to\infty} d(S(x_{nk}), x_{nk}) = p$. Thus,

$\lim_{n\to\infty}(T(y_{nk}), y_{nk}) = p$

Now, letting $k \to \infty$ in (2.8), we have $d(y, T(y)) = p = d(A, B)$.

Similiarly it can be proved that $d(x, S(x)) = d(A, B)$

In the following Theorem, the distance of A and B is obtained by considering the pair mapping (S, T ) in a regular cone metric space.

**Theorem 2.13.** Let $\varphi$ be a ultra altering distance function, $F \in C$ and $S, T : A \cup B \to A \cup B$ such that $S(A) \subset B$, $T(B) \subset A$ satisfying

$d(S(x), T(y)) - p < F(d(x, y) - p, \varphi(d(x, y) - p))$,

for all $(x, y) \in A \times B$ with $p < d(x, y)$, where p is lower bound for d on $A \times B$. Then $d(A, B) = p$.

Proof. Let $x_n$ and $y_n$ be as follows $x_{n+1} = T y_n$, $S(x_n) = y_{n+1}$ for some $(x_0, y_0) \in A\times B$, $n \in N$.

Also let $d_{n+1} = d(x_{n+1}, y_{n+1})$, we have

$d_{n+1-p} < F(d_{n-p}, \varphi(d_{n-p})) \leq d_{n-p}$.

By regularity of the cone P , we have $d_{n+1} \leq d_n$. Hence, there exists $q \in P$ such that $\lim_{n\to\infty} d_n = q$. Then $p \leq q$. Put $s_n = d_{n-p}$. Since, $s_n > 0$. we have

$s_{n+1} < F(s_n, \varphi(s_n))$. (2.9)

By regularity of the cone P , we have $d_{n+1} \leq d_n$. Hence, there exists $q \in P$ such that $\lim_{n\to\infty} d_n = q$. Then $p \leq q$. Put $s_n = d_n - p$. Since, $s_n > 0$. we have

$s_{n+1} < F(s_n, \varphi(s_n))$.

By lemma (1.10), $\lim_{n\to\infty} s_n = 0$. Thus, $\lim_{n\to\infty} d_n = p$ and so $d(A, B) = p = q$.

With choice $F(s, t) = \Psi(s)$, where $\Psi : P \to P$ is a cone L 1 −function, in Theorem(2.13) we have the following corollary.

**Corollary 2.14.** Let $\Psi : P \to P$ be a cone L 1 −function, $\varphi$ is ultra altering distance function, $F \in C$ and $S, T : A \cup B \to A \cup B$ such that $S(A) \subset B$, $T(B) \subset A$ satisfying

$d(S(x), T(y)) - p < \Psi(d(x, y) - p)$

for all (x, y) ∈ A × B with p < d(x, y), where p is lower bound for d on A × B. Then d(A, B) = p.

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