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A Study on Recognizability of Max Weighted Finite State State Automaton

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ABSTRACT

This paper deals with max weighted finite state automata (mwfa), which corresponds to the semiring $W = ([0,\infty),\cdot,\max,1,0)$. Such automata have transition with weights in W. The concept of mwfa has been outlined with some basic operations such as union, intersection, concatenation, and Kleene's closure.

Keywords: weighted automaton, Recognizable, weighted regular language.

1. INTRODUCTION

Mathematical model in classical computation, automata has been an important area in theoretical computer science[2]. It started from a seminal paper of Kleene[3], and within a few years developed into a rich mathematical research topic. The theory of computation deals with the computational logic with respect to simple machines, termed as automata. It is the study of abstract computer devices or intangible machines. Warren McCulloch and Walter Pitts, two neurophysiologists, were the first to present a description of finite automata in 1943. Finite automata played a crucial role in the theory of programming languages, complier constructions, switching circuit designing computer controller, neuron net, text editor and lexical analyzer[1].

Weighted automata are classical finite automata in which the transition carries weights. The starting point of weighted automaton is to determine the number of ways a word can be accepted or the amount of resources used for this.

The principle of weighted automata is to consider non-deterministic automata, which takes value in semiring. Max weighted finite automaton belong to the wider family of weighted automaton, as introduced by schutzenberger[6].

In this paper language related to max weighted finite state automaton is introduced. Some of the operations of max weighted finite state automata languages are considered.

2. WEIGHTED REGULAR LANGUAGES

This section introduces the notions of mwfa and a set recognized by a mwfa.

Definition 2.1.[4] A Max weighted finite state automaton (mwfa) is a six tuple $M = (Q, \Sigma, W, \mu, i, f)$, where

(i) Q is a finite non empty set of states.

(ii) Σ is a finite non-empty set of input symbols.

(iii) W is a weighting space. i.e., weighting space $W = ([0,\infty), \cdot, \max)$ where $\cdot$ is usual multiplication.

(iv) μ is a weighting function such as $\mu : Q \times \Sigma \times Q \rightarrow [0,\infty)$ is called a state transition function. The value $\mu(p,a,q)$ of $p,a,q) \in Q \times \Sigma \times Q$ represents the weighted transition from state p to state q when the input symbol is *a*.

(v) i is an initial distribution function, where $i : Q \rightarrow [0,\infty)$.

(vi) f is a final distribution function, where $f : Q \rightarrow [0,\infty)$.

Definition 2.2. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa, the extended weighting transition function for M is the weighted subset $\mu^* : Q \times \Sigma^* \times Q \rightarrow [0,\infty)$ has been defined as follows:

$\forall p, q \in Q, a \in \Sigma, x \in \Sigma^*$

$$\mu(p,\lambda,q) = \begin{cases} 1, & if \quad p = q \\ 0, & if \quad p \neq q \end{cases}$$

$$\mu^*(p, xa, q) = \max_{r \in Q} \left\{ \mu^*(p,x,r) \cdot \mu(r,a,q) \right\}$$

Definition2.3. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. Let $x \in \Sigma^*$.Then x is said to be recognized by M if

$$L(x) = \max_{p,q \in Q} \left\{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \right\} > 0$$

$$= \max_{p,q \in Q} \left\{ i(p) \cdot \left\{ \max_{r \in Q} \left(\mu^*(p,y,r) \cdot \mu(r,a,q) \right) \right\} \cdot f(q) \right\}$$

Where $x = ya$.

Lemma 2.4. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. Let $x \in \Sigma^*$. Then x is recognized if and only if there exists $p, q \in Q$ such that $w(x) = \left\{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \right\} > 0$

Proof: Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. Suppose $x \in \Sigma^*$ is recognized. Implies that $L(x) = \max_{p,q \in Q} \{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \} > 0$.

Since Q is finite, and each term is non-negative, there exists $p, q \in Q$ such that $w(x) = \{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \} > 0$.

Conversely, suppose $p, q \in Q$ such that $i(p) \cdot \mu^*(p,x,q) \cdot f(q) > 0$.

Now, $i(p) \cdot \mu^*(p,x,q) \cdot f(q) \le \max_{p,q \in Q} \{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \}$

Therefore $L(x) > 0$. Hence x is recognized.

Theorem 2.5. Let $M = (Q, \Sigma, W, \mu, i, f)$ be the mwfa .Then

$$\mu^*(p, xy, q) = \max_{s \in Q} \{ \mu^*(p,x,s) \cdot \mu^*(s,y,q) \} \quad \forall p, q \in Q, \& \forall x, y \in \Sigma^*.$$

Proof: Let $p, q \in Q$ and $x, y \in \Sigma^*$.

We prove the result by induction on $|y| = n$. For $n = 0$ then $y = \lambda$, the result is obvious.

Suppose the result is true for all $u \in \Sigma^*$ such that $|u| = n-1, n > 0$.

Let $y = ua$, where $u \in \Sigma^*, a \in \Sigma$ and $|u| = n-1, n > 0$.

Now

$$\mu^*(p, xy, q) = \max_{r \in Q} \{ \mu^*(p, xu, r) \cdot \mu(r, a, q) \}$$

$$= \max_{s \in Q} \left\{ \mu^*(p,x,s) \cdot \left(\max_{r \in Q} \{ \mu^*(s,u,r) \cdot \mu(r,a,q) \} \right) \right\}$$

$$= \max_{s \in Q} \{ \mu^*(p,x,s) \cdot \mu^*(s,y,q) \}$$

Thus the result is true for $|y| = n$ and hence the result.

Example 2.6. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa , where $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, $\mu : Q \times \Sigma \times Q \to [0, \infty)$ is defined as follows:

$\mu(q_1, a, q_1) = 2$ $\quad$ $\mu(q_2, a, q_3) = 6$ $\quad$ $\mu(q_1, b, q_2) = 3$

$\mu(q_1, b, q_1) = 4$ $\quad$ $\mu(q_3, a, q_3) = 2.5$ $\quad$ $\mu(q_1, a, q_2 = 5$

$\mu(q_3, b, q_3 = 3.5$

We omit the weighted values which are zero.

$i : Q \to [0,\infty)$ is defined by $i(q_1) = 2, \quad i(q_2) = 3.$

$f : Q \to [0,\infty)$ is defined by $f(q_3) = 5.$

The transition diagram is shown below:

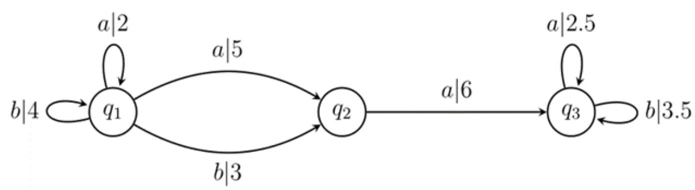

Figure 2.1

The language accepted by *M* is a weighted subset $L : \Sigma^* \to [0,\infty)$ such that

$$L(x) = \begin{cases} w_1, & w_1 \geq 180 \quad if \quad x \in \{a,b\}^* ba \{a,b\}^* \\ w_2, & w_2 \geq 300 \quad if \quad x \in \{a,b\}^* aa \{a,b\}^* \\ w_3, & w_3 \geq 90 \quad if \quad x \in a\{a,b\}^* \\ 0, & otherwise \end{cases}$$

Now for every incomplete mwfa M , there exist an equivalent complete mwfa M^C such that the language accepted by both are same is proved with illustrations.

Definition 2.7. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. *M* is called complete if for all $p \in Q, a \in \Sigma,$

there exists $q \in Q$ such that $\mu(p,a,q) > 0.$

Theorem 2.8. Let $M = (Q, \Sigma, W, \mu, i, f)$ be an incomplete mwfa, then there exist a mwfa M^C which is the completion of *M* such that, the weighted regular language accepted by *M* and M^C are equal.

Proof: Let $M = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, i, f)$ be an incomplete mwfa and the weighted language accepted by it be L . Let $Q^c = Q \cup \{t\}$ where t is a new state such that $t \notin Q$. $\forall p \in Q$, let $0 < m_p < \infty$ and let $0 < m < \infty$. Define $M^C = (\mathrm{Q}^C, \Sigma, \mathrm{W}, \mu^C, i^C, f^C)$ where $\mu^c : \mathrm{Q}^c \times \Sigma \times \mathrm{Q}^c \to [0,\infty)$ is defined as follows: $\forall p, q \in Q, a \in \Sigma$

(i) $\mu^c(p,a,q) = \mu(p,a,q), \quad if \quad \mu(p,a,q) > 0$

(ii) $$\mu^c(p,a,t) = \begin{cases} m_p, & if \quad \max\limits_{q \in Q} \{\mu(p,a,q)\} = 0 \\ 0, & if \quad \max\limits_{q \in Q} \{\mu(p,a,q)\} > 0 \end{cases}$$

(iii) $\mu^c(t,a,\mathrm{p}) = \begin{cases} m, & if \quad p = t \\ 0, & if \quad p \neq t \end{cases}$

$i^c : Q^c \to [0,\infty)$ is defined by $i^c(p) = \begin{cases} i(\mathrm{p}), & if \quad p \in Q \\ 0, & if \quad p \notin Q \end{cases}$

$f^c : Q^c \to [0,\infty)$ is defined by $f^c(p) = \begin{cases} f(p), & if \quad p \in Q \\ 0, & if \quad p \notin Q \end{cases}$

Clearly M^C is a complete mwfa. Let L_1 be a weighted language accepted by M^C.

To prove $L_1(x) = L(x)$ for all $x \in \sum^*$

Case (i) : If $\mathrm{L}_1(\mathrm{x}) = 0.$

$$L_1(x) = \max_{p,q \in Q^C} \left\{ i^C(p) \cdot \mu^{C^*}(p,x,q) \cdot f^C(q) \right\} = 0$$

Implies that $i^C(\mathrm{p}) = 0$, or $\mu^{C*}(\mathrm{p},x,q) = 0$,or $f^C(q) = 0, \forall p,q \in Q^C.$

From the definition of M^C, we have $L(x) = 0.$

Case (ii): If $L_1(x) > 0.$

Since Q^C is finite, x is finite length , $L_1(x)$ is finite. Consider any term in $L_1(x)$. Let it be $w_1(x).$ $w_1(x) = i^C(p) \cdot \mu^{C*}(\mathrm{p,x},q) \cdot f^C(q) > 0.$

$i^C(p) > 0$, implies that $i\ (p) > 0$, and $p \in Q$. $f^C(q) > 0$, implies that $f\ (q) > 0$, and $q \in Q.$

$\mu^{C*}(\mathrm{p,x},q) > 0, \& f^C(q) > 0$ implies that $q \neq t$. Therefore M^C never enters into the dead state t in the sequence of moves. Therefore $\mu^{C*}(\mathrm{p},x,q) = \mu^*(\mathrm{p},x,q).$ we have $w_1(x) = w(x).$

Then $\max\limits_{p,q \in Q^C} \left\{ i^C(p) \cdot \mu^{C^*}(p,x,q) \cdot f^C(q) \right\} = \max\limits_{p,q \in Q} \left\{ i(p) \cdot \mu^*(p,x,q) \cdot f(q) \right\}$

i.e., $L_1(x) = L(x) \forall x \in \sum^*$. Similarly we have $L(x) = L_1(x) \forall x \in \sum^*$.

3. SOME OPERATIONS ON WEIGHTED REGULAR LANGUAGES

Weighted regular languages with some closure properties such as union, intersection, concatenation and Kleene's closure are defined in this section.

Theorem 3.1. Let $M_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \mathrm{i}_1, \mathrm{f}_1)$ and $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be two mwfa's with weighted regular languages L_1 and L_2 respectively. Then L is a weighted regular language accepted by $\mathrm{M}_1 \cup M_2$, where $L = L_1 \cup L_2$ and $L(x) = \max\left\{ L_1(x), L_2(x) \right\}$.

Proof: Let $M_1=(Q_1,\Sigma,W,\mu_1,i_1,f_1)$ and $M_2=(Q_2,\Sigma,W,\mu_2,i_2,f_2)$ be two mwfa's and $Q_1\cap Q_2=\varnothing$. The union of M_1 and M_2 is a mwfa $M=M_1\cup M_2=(Q',\Sigma,W,\mu,i,f)$ where

$Q'=Q_1\cup Q_2\cup\{q_0\}=Q\cup\{q_0\}$, $\mu:Q'\times\Sigma\times Q'\to[0,\infty)$ is defined as follows:

(i) $\mu(p,a,q)=\mu_1(p,a,q),\forall\, p,q\in Q_1, a\in\Sigma$

(ii) $\mu(p,a,q)=\mu_2(p,a,q),\forall\, p,q\in Q_2, a\in\Sigma$

(iii) For $p,r\in Q_1$, if $i_1(p)>0$ and $\mu_1(p,a,r)>0$ $\quad\exists q_0\notin Q$ such that $\mu(q_0,a,r)=i_1(p)\cdot\mu_1(p,a,r)$

(iv) For $p,r\in Q_2$, $i_2(p)>0$ and $\mu_2(p,a,r)>0$, $\quad\exists q_0\notin Q$ such that $\mu(q_0,a,r)=i_2(p)\cdot\mu_2(p,a,r)$

$i:Q'\to[0,\infty)$ is defined by $i(p)=\begin{cases}0, & if \quad p\in Q\\ 1, & if \quad p=q_0\end{cases}$

$f:Q'\to[0,\infty)$ is defined by $f(p)=\begin{cases}f_1(p), & if \quad p\in Q_1\\ f_2(p), & if \quad p\in Q_2\\ 0, & otherwise\end{cases}$

From the definition of M ,we have for all $x\in\Sigma^*$,

$$\mu^*(p,x,q)=\begin{cases}\mu_1^*(p,x,q), & if \quad p,q\in Q_1\\ \mu_2^*(p,x,q), & if \quad p,q\in Q_2\\ 0, & otherwise\end{cases}$$

Let $x=ay\in\Sigma^*$ where $a\in\Sigma$, $y\in\Sigma^*$, then if $q\in Q_1$ and $q_0\in Q'\setminus Q$.

$$\mu^*(q_0,x,q)=\max\left\{\max_{r\in Q_1}\left\{\mu(q_0,a,r)\cdot\mu^*(r,y,q)\right\},\max_{r\in Q_2}\left\{\mu(q_0,a,r)\cdot\mu^*(r,y,q)\right\}\right\}$$

$$=\max\left\{\max_{r\in Q_1}\left\{i_1(p)\cdot\mu_1(p,a,r)\cdot\mu_1^*(r,y,q)\right\},0\right\}$$

$$=i_1(p)\cdot\max_{r\in Q_1}\left\{\mu_1(p,a,r)\cdot\mu_1^*(r,y,q)\right\}$$

$$=i_1(p)\cdot\mu_1^*(p,x,q) \tag{3.1}$$

Similarly, if $x\in\Sigma^*$, then if $q\in Q_2$ and $q_0\in Q'\setminus Q$ then

$$\mu^*(q_0,x,q)=i_2(p)\cdot\mu_2^*(p,x,q) \tag{3.2}$$

Now,

$$L(x) = \max\left\{\max_{q\in Q_1}\left\{i(q_0)\cdot\mu^*(q_0,x,q)\cdot f(q)\right\}, \max_{q\in Q_2}\left\{i(q_0)\cdot\mu^*(q_0,x,q)\cdot f(q)\right\}\right\}$$

$$= \max\left\{\max_{q\in Q_1}\left\{i(q_0)\cdot i_1(p)\cdot\mu_1^*(p,x,q)\cdot f_1(q)\right\}, \max_{q\in Q_2}\left\{i(q_0)\cdot i_2(p)\cdot\mu_2^*(p,x,q)\cdot f_2(q)\right\}\right\}$$

(from (3.1) and (3.2))

$$= \max\left\{\max_{p,q\in Q_1}\left\{i_1(p)\cdot\mu_1^*(p,x,q)\cdot f_1(q)\right\}, \max_{p,q\in Q_2}\left\{i_2(p)\cdot\mu_2^*(p,x,q)\cdot f_2(q)\right\}\right\}$$

[since $i(\mathrm{q}_0)=1$]

$$= \max\left\{L_1(x), L_2(x)\right\} \qquad \forall x\in\Sigma^*$$. Hence $L = L_1 \cup L_2$.

Example 3.2. Let $M_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \mathrm{i}_1, \mathrm{f}_1)$ and $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be two mwfa's where $\mathrm{Q}_1 = \{q_1, \mathrm{q}_2, \mathrm{q}_3\}$, $\Sigma = \{a,b\}$, $\mu_1 : Q_1 \times \Sigma \times Q_1 \to [0,\infty)$ is defined as follows:

$\mu_1(\mathrm{q}_1, \mathrm{a}, q_2) = 2.5$, $\quad\mu_1(\mathrm{q}_1, \mathrm{b}, q_2) = 3$, $\quad\mu_1(\mathrm{q}_2, \mathrm{a}, q_3) = 5$,

$\mu_1(\mathrm{q}_3, \mathrm{a}, q_3) = 6$, $\quad\mu_1(\mathrm{q}_2, \mathrm{b}, q_3) = 4$, $\quad\mu_1(\mathrm{q}_3, \mathrm{b}, q_3) = 7$.

$\mathrm{i}_1 : Q_1 \to [0,\infty)$ is defined by $\mathrm{i}_1(q_1) = 3$

$\mathrm{f}_1 : Q_1 \to [0,\infty)$ is defined by $\mathrm{f}_1(q_3) = 5$.

The language accepted by M_1 is a weighted subset $L_1 : \Sigma^* \to [0,\infty)$ such that

$$\mathrm{L}_1(\mathrm{x}) = \begin{cases} \mathrm{w}_1, & w_1 \geq 187.5 \quad if \quad x \in aa\{a,b\}^* \\ \mathrm{w}_2, & w_2 \geq 150 \quad if \quad x \in ab\{a,b\}^* \\ w_3, & w_3 \geq 225 \quad if \quad x \in ba\{a,b\}^* \\ w_4, & w_4 \geq 180 \quad if \quad x \in bb\{a,b\}^* \\ 0, & otherwise \end{cases}$$

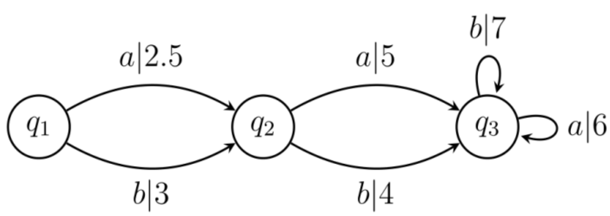

Figure 3.1

Let $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be a mwfa, where $\mathrm{Q}_2 = \{q_4, \mathrm{q}_5, \mathrm{q}_6\}$, $\Sigma = \{a,b\}$,

$\mu_2 : Q_2 \times \Sigma \times Q_2 \to [0,\infty)$ is defined as follows:

$\mu_2(q_4, a, q_5) = 3$, $\mu_2(q_4, b, q_5) = 4$, $\mu_2(q_5, b, q_5) = 5$,
$\mu_2(q_5, a, q_6) = 1$, $\mu_2(q_5, a, q_5) = 2$.
$i_2 : Q_2 \to [0, \infty)$ is defined by $i_2(q_4) = 6$
$f_2 : Q_2 \to [0, \infty)$ is defined by $f_2(q_6) = 7$.

The language accepted by M_2 is a weighted subset $L_2 : \Sigma^* \to [0, \infty)$ such that

$$L_2(x) = \begin{cases} w_1, & w_1 \geq 126 \quad if \quad x \in a\{a,b\}^* a \\ w_2, & w_2 \geq 168 \quad if \quad x \in b\{a,b\}^* a \\ 0, & otherwise \end{cases}$$

The transition diagram is shown below:

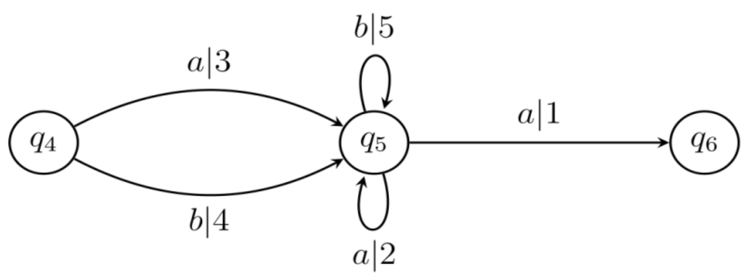

Figure 3.2

Let $M = M_1 \cup M_2 = (Q', \Sigma, W, \mu, i, f)$ be the mwfa where,
$Q' = Q_1 \cup Q_2 \cup \{q_0\}$, $\Sigma = \{a, b\}$,
$\mu : Q' \times \Sigma \times Q' \to [0, \infty)$ is defined as follows:

$\mu(q_0, a, q_2) = 7.5$, $\mu(q_0, b, q_2) = 9$, $\mu(q_0, a, q_5) = 18$,
$\mu(q_0, b, q_5) = 24$, $\mu(q_1, a, q_2) = 2.5$, $\mu(q_1, b, q_2) = 3$,
$\mu(q_2, a, q_3) = 5$, $\mu(q_3, a, q_3) = 6$, $\mu(q_2, b, q_3) = 4$,
$\mu(q_3, b, q_3) = 7$, $\mu(q_4, a, q_5) = 3$, $\mu(q_4, b, q_5) = 4$,
$\mu(q_5, b, q_5) = 5$, $\mu(q_5, a, q_6) = 1$, $\mu(q_5, a, q_5) = 2$.
$i : Q' \to [0, \infty)$ is defined by $i(q_0) = 1$
$f : Q' \to [0, \infty)$ is defined by $f(q_3) = 5$, $f(q_6) = 7$.

The transition diagram is shown below:

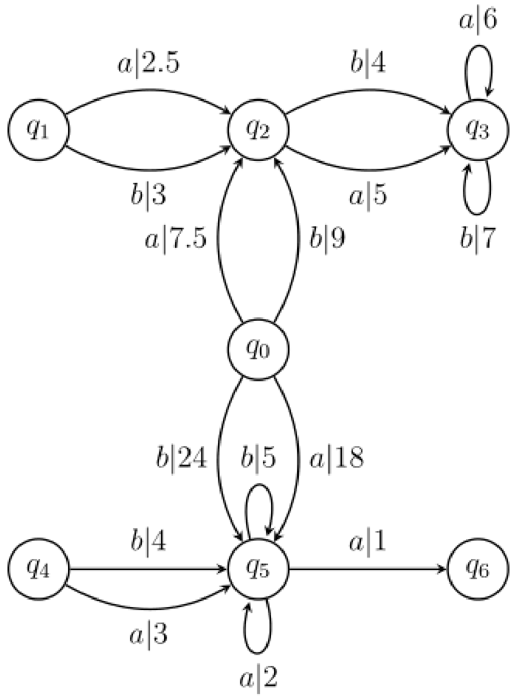

Figure 3.3: Union of two mwfa.

The language accepted by M is a weighted subset $L:\Sigma^* \to [0,\infty)$ such that

$$L(x)=\begin{cases} w_1, & w_1 \geq 150 \quad if \quad x \in ab\{a,b\}^* \\ w_2, & w_2 \geq 225 \quad if \quad x \in ba\{a,b\}^* \\ w_3, & w_3 \geq 126 \quad if \quad x \in a\{a,b\}^* a \\ w_4, & w_4 \geq 168 \quad if \quad x \in b\{a,b\}^* a \\ w_5, & w_5 \geq 187.5 \, if \quad x \in aa\{a,b\}^* \\ w_6, & w_6 \geq 180 \quad if \quad x \in bb\{a,b\}^* \\ 0, & otherwise \end{cases}$$

Definition 3.3. Let $M_1 = (Q_1, \Sigma, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, W, \mu_2, i_2, f_2)$ be two mwfa's $Q_1 \cap Q_2 = \emptyset$. Then the mwfa $M_1 \cup M_2 = (Q_1 \times Q_2, \Sigma, W, \mu_1 \times \mu_2, i_1 \times i_2, f_1 \times f_2)$, where $\mu_1 \times \mu_2 : (Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2) \to [0,\infty)$ is defined by $(\mu_1 \times \mu_2)((p_1, p_2), a, (q_1, q_2)) = \mu_1(p_1, a, q_1) \cdot \mu_2(p_2, a, q_2)$,

$(\mathrm{i}_1 \times i_2) : Q_1 \times Q_2 \to [0,\infty)$ is defined by $(\mathrm{i}_1 \times i_2)(p_1, p_2) = \mathrm{i}_1(p_1) \cdot \mathrm{i}_2(p_2)$
$(f_1 \times f_2) : Q_1 \times Q_2 \to [0,\infty)$ is defined by $(f_1 \times f_2)(p_1, p_2) = f_1(p_1) \cdot f_2(p_2)$.

Lemma 3.4. Let $M_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \mathrm{i}_1, \mathrm{f}_1)$ and $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be two mwfa's. Then, $(\mu_1 \times \mu_2)^*((p_1, p_2), x, (q_1, q_2)) = \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, x, q_2)$, $\forall\, \mathrm{x} \in \Sigma^*$, $(p_1, p_2), (\mathrm{q}_1, q_2) \in Q_1 \times Q_2$.

Proof: We prove the result by induction on $|x| = n$, $x \in \Sigma^*$.
Let $((p_1, p_2), x, (q_1, q_2)) \in (Q_1 \times Q_2) \times \Sigma^* \times (Q_1 \times Q_2)$. For $n = 0, x = \lambda$ the result is obvious.
Suppose the result is true $\forall x \in \Sigma^*, |x| < n$. Let $x = ya, y \in \Sigma^*, a \in \Sigma, |y| = n-1$.
Now, $(\mu_1 \times \mu_2)^*((p_1, p_2), \mathrm{ya}, (\mathrm{q}_1, q_2))$

$$= \max_{(\mathrm{r}_1, r_2) \in (Q_1 \times Q_2)} \left\{ \mu_1^*(\mathrm{p}_1, \mathrm{y}, \mathrm{r}_1) \cdot \mu_2^*(\mathrm{p}_2, \mathrm{y}, \mathrm{r}_2) \cdot \mu_1(\mathrm{r}_1, \mathrm{a}, \mathrm{q}_1) \cdot \mu_2(\mathrm{r}_2, \mathrm{a}, \mathrm{q}_2) \right\}$$
$$= \left\{ \max_{r_1 \in Q_1} \left\{ \mu_1^*(\mathrm{p}_1, \mathrm{y}, \mathrm{r}_1) \cdot \mu_1(\mathrm{r}_1, \mathrm{a}, \mathrm{q}_1) \right\} \cdot \max_{r_2 \in Q_2} \left\{ \mu_2^*(\mathrm{p}_2, \mathrm{y}, \mathrm{r}_2) \cdot \mu_2(\mathrm{r}_2, \mathrm{a}, \mathrm{q}_2) \right\} \right\}$$
$$= \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, x, q_2)$$

This completes the lemma.

Theorem 3.5. Let $M_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \mathrm{i}_1, \mathrm{f}_1)$ and $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be two mwfa's.
with L_1 and L_2 as a weighted regular languages respectively. Then L is a weighted regular language accepted by mwfa M such that $\mathrm{L}_1 \cap L_2$ and $\mathrm{L(x)} = L_1(x) \cdot L_2(x)$.

Proof: Let $M = M_1 \cap M_2$ and L be a weighted regular language accepted by M. Let $x \in \Sigma^*$

$$L(x) = \max_{(p_1, p_2), (q_1, q_2) \in (Q_1, Q_2)} \left\{ (i_1 \times i_2)(p_1, p_2) \cdot (\mu_1 \times \mu_2)^*((p_1, p_2), x, (q_1, q_2)) \cdot (f_1 \times f_2)(q_1, q_2) \right\}$$
$$= \max_{(p_1, p_2), (q_1, q_2) \in (Q_1, Q_2)} \left\{ i_1(p_1) \cdot i_2(p_2) \cdot \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, x, q_2) \cdot f_1(q_1) \cdot f_2(q_2) \right\}$$
$$= \max_{(p_1, q_1) \in Q_1} \left\{ i_1(p_1) \cdot \mu_1^*(p_1, x, q_1) \cdot f_1(q_1) \right\} \cdot \max_{(p_2, q_2) \in Q_2} \left\{ i_2(p_2) \cdot \mu_2^*(p_2, x, q_2) \cdot f_2(q_2) \right\}$$

(by previous lemma)

$= L_1(x) \cdot L_2(x)$. Thus $L(x) = L_1(x) \cdot L_2(x)$ for all $x \in \Sigma^*$ and $L = L_1 \cap L_2$.

Theorem 3.6. Let A and B be a recognizable set over Σ^* with weighted regular languages L_1 and L_2, which is accepted by mwfa's $M_!$ and M_2 respectively. Then the set AB is recognizable by a mwfa M such that $\mathrm{L} = \mathrm{L}_1 L_2$.

Proof: Let $M_1 = (Q_1, \Sigma, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, W, \mu_2, i_2, f_2)$ be two mwfa's , $Q_1 \cap Q_2 = \emptyset$ with weighted regular languages be L_1 and L_2 respectively.

i.e., $L_1 : \Sigma^* \to [0,\infty)$ such that $L_1(x) > 0,\ \forall\ x \in A$.

$L_2 : \Sigma^* \to [0,\infty)$ such that $L_2(x) > 0,\ \forall\ x \in B$.

Define mwfa $M = (Q, \Sigma, W, \mu, i, f)$, where $Q = Q_1 \cup Q_2$, $\mu : Q \times \Sigma \times Q \to [0,\infty)$ is defined as follows:

(i) $\forall\ p, q \in Q_1, a \in \Sigma, \mu(p,a,q) = \mu_1(p,a,q)$

(ii) $\forall\ p, q \in Q_2, a \in \Sigma, \mu(p,a,q) = \mu_2(p,a,q)$

(iii) $\forall\ p \in Q_1, \forall q \in Q_2$ and if $f_1(p) > 0$, $r \in Q_2$ with $i_2(r) > 0$ and $\mu_2(r,a,q) > 0$ then

$$\mu(p,a,q) = \begin{cases} f_1(p) \cdot i_2(r) \cdot \mu_2(r,a,q) \\ 0, \quad otherwise \end{cases}$$

$i : Q \to [0,\infty)$ is defined by

$$i(p) = \begin{cases} i_1(p) \quad if \quad p \in Q_1, \\ 0, \quad otherwise \end{cases}$$

$f : Q \to [0,\infty)$ is defined by

$$f(p) = \begin{cases} f_2(p) \quad if \quad p \in Q_2, \\ 0, \quad otherwise \end{cases}$$

Let $L(z) = \max_{p,q \in Q} \left\{ i(p) \cdot \mu^*(p,z,q).f(q) \right\} > 0$ (3.3)

Since Q is finite and z is of finite length. Therefore is finite. From the definition of M there exists $x = a_1a_2...a_n \in A$, $y = b_1b_2...b_m \in B$, $z = xy$ & $p_0, p_1, ...p_n, q_1, q_2, ...q_n \in Q$.

Consider any term in $L(z)$. Let it be $w(xy)$ where $x \in A, y \in B$.

$$\begin{aligned} w(xy) &= i(p_0) \cdot \mu^*(p_0, xy, q_m) \cdot f(q_m) \\ &= i(p_0) \cdot \mu(p_0,a_1,p_1) \cdot \mu(p_1,a_2,p_2) \cdots \mu(p_{n-1},a_n,p_n) \cdot \mu(p_n,b_1,q_1) \cdot \mu(q_1,b_2,q_2) \cdots \\ &\quad \mu(q_{m-1},b_m,q_m) \cdot f(q_m) \end{aligned} \quad (3.4)$$

From the definition of M we have,

$(i) \mu(p_{i-1}, a_i, p_i) = \mu_1(p_{i-1}, a_i, p_i)$. Where $i = 1, 2, ...n$.

$(ii) \mu(q_{i-1}, b_i, q_i) = \mu_2(q_{i-1}, b_i, q_i)$. where $i = 2, 3, ...m$

(iii)Since $\mu(p_n, b_1, q_1) > 0$, we have

$\mu(p_n, b_1, q_1) = f_1(p_n) \cdot i_2(q_0) \cdot \mu_2(q_0, b_1, q_1) > 0$.

$(3.4) \Rightarrow w(xy) = i_1(p_0)\cdot\mu_1(p_0,a_1,p_1)\cdot\mu_1(p_1,a_2,p_2)\cdots\mu_1(p_{n-1},a_n,p_n)\cdot f_1(p_n)\cdot i_2(q_0)\cdot\mu_1(q_0,b_1,q_1)\cdot$

$\mu_2(q_1,b_2,q_2)\cdots\mu_2(q_{m-1},b_m,q_m)\cdot f_2(q_m)$

$= w_1(x)\cdot w_2(y)$. Thus $w(xy) = w_1(x)\cdot w_2(y)$.

This is true for every term in (3.3) Then,

$$\max_{p_o,q_m\in Q}\left\{i(p_0)\cdot\mu^*(p_0,z,q_m).f(q_m)\right\} = \left\{\max_{p_0,p_n\in Q_1}\left\{i_1(p_o)\cdot\mu_1^*(p_0,x,p_n).f_i(p_n)\right\}\right\}\cdot$$

$$\left\{\max_{q_o,q_m\in Q_2}\left\{i_2(q_0)\cdot\mu_2^*(q_o,y,q_m)\cdot f_2(q_m)\right\}\right\}$$

$$L(z) = L_1(x)\cdot L_2(y).$$

Let $x\in A$ & $y\in B$. Therefore $L_1(x)>0$ and $L_2(y)>0$, then

$$L_1(x) = \max_{p_0,p_n\in Q_1}\left\{i_1(p_0)\cdot\mu_1^*(p_0,x,p_n)\cdot f_1(p_n)\right\}$$

$L_1(x)$ is finite. Now, consider any tern in $L_1(x)$, let it be $w_1(x)$.

$$w_1(x) = i_1(p_0)\cdot\mu_1(p_0,a_1,p_1)\cdot\mu_1(p_1,a_2,p_2)\cdots\mu_1(p_{n-1},a_n,p_n)\cdot f_1(p_n)$$

where $i_1(p_0)>0$, $f_1(p_n)>0$, $x = a_1a_2...a_n\in A$, $p_0, p_1....p_n\in Q_1$

Similarly $w_2(y) = i_2(q_0)\cdot\mu_2(q_0,b_1,q_1)\cdot\mu_2(q_1,b_2,q_2)\cdots\mu_2(q_{m-1},b_m,q_m)\cdot f_2(q_m)$

where $i_2(q_0)>0$, $f_2(q_m)>0$, $y = b_1b_2...b_m\in B$, $q_0,q_1....q_m\in Q_2$

$$w_1(x)\cdot w_2(y) = \left\{i_1(p_0)\cdot\mu_1(p_0,a_1,p_1)\cdot\mu_1(p_1,a_2,p_2)\cdots\mu_1(p_{n-1},a_n,p_n)\cdot f_1(p_n)\right\}\cdot$$

$$\left\{i_2(q_0)\cdot\mu_2(q_0,b_1,q_1)\cdot\mu_2(q_1,b_2,q_2)\cdots\mu_2(q_{m-1},b_m,q_m)\cdot f_2(q_m)\right\}$$

From the definition of M, we have

$w_1(x)\cdot w_2(y) = w(xy)$. Then

$$\left\{\max_{p_0,p_n\in Q_1}\left\{i_1(p_o)\cdot\mu_1^*(p_0,x,p_n).f_i(p_n)\right\}\right\}\cdot\left\{\max_{q_o,q_m\in Q_2}\left\{i_2(q_0)\cdot\mu_2^*(q_o,y,q_m)\cdot f_2(q_m)\right\}\right\}$$

$$= \max_{p_o,q_m\in Q}\left\{i(p_0)\cdot\mu^*(p_0,xy,q_m).f(q_m)\right\}$$

Thus $L_1(x)\cdot L_2(y) = L(xy) = L(z)$.

Example 3.7. Let M_1 be a mwfa, where $Q_1 = \{q_1,q_2,q_3\}$, $\Sigma = \{a,b\}$,

$\mu_1 : Q_1\times\Sigma\times Q_1\to[0,\infty)$ is defined as follows:

$\mu_1(q_1,a,q_2) = 2$, $\mu_1(q_1,b,q_2) = 3$, $\mu_1(q_2,a,q_3) = 6$,

$\mu_1(q_3,b,q_3) = 8$, $\mu_1(q_2,b,q_3) = 4$, $\mu_1(q_3,a,q_3) = 7$.

$i_1 : Q_1\to[0,\infty)$ is defined by $i_1(q_1) = 4$,

$f_1 : Q_1\to[0,\infty)$ is defined by $f_1(q_3) = 5$.

The transition diagram is shown below:

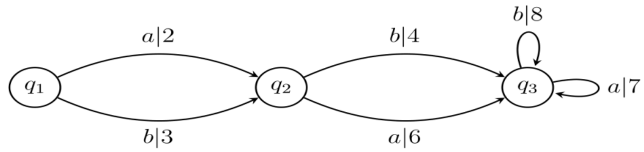

Figure 3.5

The language accepted by M_1 is a weighted subset $L_1 : \Sigma^* \to [0,\infty)$ such that

$$L_1(x) = \begin{cases} w_1, & w_1 \geq 96 \quad if \quad x \in ab\{a,b\}^* \\ w_2, & w_2 \geq 216 \quad if \quad x \in ba\{a,b\}^* \\ \mathrm{w}_3, & w_3 \geq 144 \quad if \quad x \in aa\{a,b\}^* \\ \mathrm{w}_4, & w_4 \geq 144 \quad if \quad x \in bb\{a,b\}^* \\ w_5, & w_5 \geq 24 \quad if \quad x \in b\{a,b\}^* \\ w_6, & w_6 \geq 36 \quad if \quad x \in a\{a,b\}^* \\ 0, & otherwise \end{cases}$$

Let $M_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \mathrm{i}_2, \mathrm{f}_2)$ be a mwfa, where $\mathrm{Q}_2 = \{q_4, \mathrm{q}_5, \mathrm{q}_6\}$, $\Sigma = \{a,b\}$,

$\mu_2 : Q_2 \times \Sigma \times Q_2 \to [0,\infty)$ is defined as follows:

$\mu_2(\mathrm{q}_4, \mathrm{a}, q_5) = 6$, $\mu_2(\mathrm{q}_4, \mathrm{b}, q_5) = 4$, $\mu_2(\mathrm{q}_6, \mathrm{a}, q_6) = 8$,

$\mu_2(\mathrm{q}_5, \mathrm{b}, q_6) = 7$, $\mu_2(\mathrm{q}_6, \mathrm{b}, q_6) = 9$.

$\mathrm{i}_2 : Q_2 \to [0,\infty)$ is defined by $\mathrm{i}_2(q_4) = 3$

$\mathrm{f}_2 : Q_2 \to [0,\infty)$ is defined by $f_2(q_6) = 4$

The language accepted by M_2 is a weighted subset $L_2 : \Sigma^* \to [0,\infty)$ such that

$$L_2(x) = \begin{cases} w_1, & w_1 \geq 336 \quad if \quad x \in bb\{a,b\}^* \\ w_2, & w_2 \geq 504 \quad if \quad x \in ab\{a,b\}^* \\ 0, & otherwise \end{cases}$$

The transition diagram is shown below:

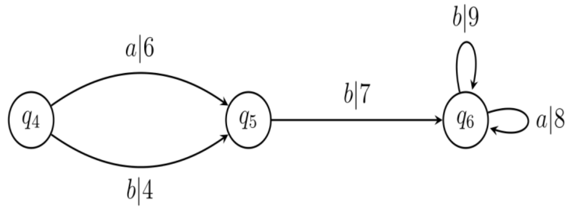

Figure 3.6

Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa, where $Q = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $\Sigma = \{a, b\}$,
$\mu : Q \times \Sigma \times Q \to [0, \infty)$ is defined as follows:

$\mu(q_1, a, q_2) = 2$, $\mu(q_1, b, q_2) = 3$, $\mu(q_2, b, q_3) = 4$,
$\mu(q_3, a, q_3) = 7$, $\mu(q_3, a, q_5) = 54$, $\mu(q_4, b, q_5) = 4$,
$\mu(q_4, a, q_5) = 6$, $\mu(q_5, b, q_6) = 7$, $\mu(q_6, b, q_6) = 9$,
$\mu(q_2, a, q_3) = 6$, $\mu(q_6, a, q_6) = 8$, $\mu(q_3, b, q_3) = 8$,
$\mu(q_3, b, q_5) = 36$.

$i : Q \to [0, \infty)$ is defined by $i(q_1) = 4$, $i(q_2) = 2$

$f : Q \to [0, \infty)$ is defined by $f(q_6) = 3$

The transition diagram is shown below:

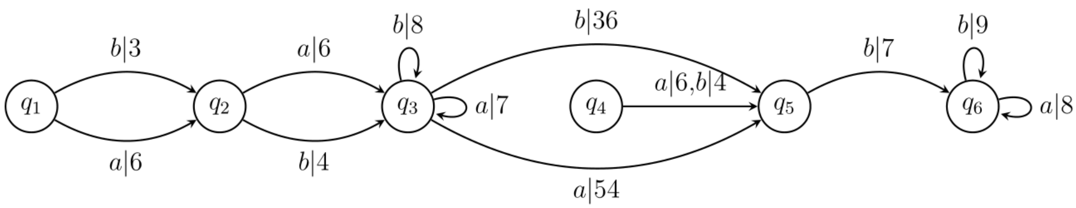

Figure 3.7

The language accepted by M is a weighted subset $L : \Sigma^* \to [0, \infty)$ such that

$$L(x) = \begin{cases} w_1, & w_1 \geq 72{,}576 \quad if \quad x \in ba\{a,b\}^* bb\{a,b\}^* \\ w_2, & w_2 \geq 48{,}384 \quad if \quad x \in bb\{a,b\}^* bb\{a,b\}^* \\ w_3, & w_3 \geq 48{,}384 \quad if \quad x \in aa\{a,b\}^* bb\{a,b\}^* \\ w_4, & w_4 \geq 32{,}256 \quad if \quad x \in ab\{a,b\}^* bb\{a,b\}^* \\ w_5, & w_5 \geq 108{,}864 \quad if \quad x \in ba\{a,b\}^* ab\{a,b\}^* \\ w_6, & w_6 \geq 72{,}576 \quad if \quad x \in bb\{a,b\}^* ab\{a,b\}^* \\ w_7, & w_7 \geq 72{,}576 \quad if \quad x \in aa\{a,b\}^* ab\{a,b\}^* \\ w_8, & w_8 \geq 48{,}384 \quad if \quad x \in ab\{a,b\}^* ab\{a,b\}^* \\ w_9, & w_9 \geq 12{,}096 \quad if \quad x \in a\{a,b\}^* bb\{a,b\}^* \\ w_{10}, & w_{10} \geq 18{,}144 \quad if \quad x \in a\{a,b\}^* ab\{a,b\}^* \\ w_{11}, & w_{11} \geq 8{,}064 \quad if \quad x \in b\{a,b\}^* bb\{a,b\}^* \\ w_{12}, & w_{12} \geq 12{,}096 \quad if \quad x \in b\{a,b\}^* ab\{a,b\}^* \\ 0, & otherwise \end{cases}$$

If $x \in ba$, then $L_1(ba) = \max\{216, 168\} = 216$

If $y = ab$, then $L_2(ab)$ =504.

If $z = baab$, then $L(baab) = 108,864$

$L_1(ba) L_2(ab) = 216 \cdot 504 = 108,864$. Thus $L(baab) = L_1(ba) \cdot L_2(ab)$.

Theorem 3.8. Let $A \subseteq \sum^*$ be recognizable set with weighted regular languages L_1, which is accepted by a mwfa M_1. Then A^* is recognizable with weighted regular languages L, accepted by a mwfa *M* such that $L = L^*_1$.

Proof: Let $M_1 = (Q_1, \sum, W, \mu_1, i_1, f_1)$ be a mwfa with L_1 being the language accepted by it. We have $L_1(x) > 0, \forall x \in A$.

Define mwfa $M = (Q, \sum, W, \mu, i, f)$ with weighted language L, where $Q = Q_1$,

$\mu : Q \times \sum \times Q \to [0, \infty)$ is defined as follows :

(i) $\mu(p, a, q) = \mu_1(p, a, q) \quad \forall p, q \in Q_1, \quad a \in \sum$

(ii) For $p, q \in Q$, if $r \in Q_1$, $i_1(q) > 0$, $f_1(r) > 0$, & $\mu_1(p, a, r) > 0$ then include

$$\mu(p, a, q) = \mu_1(p, a, r) \cdot f_1(r) \cdot i_1(q)$$

$i : Q \to [0, \infty)$ is defined by $i(q) = i_1(q) \ \forall q \in Q_1$,

$f : Q \to [0, \infty)$ is defined by $f(q) = f_1(q) \ \forall q \in Q_1$

Let $z \in A^*$, $L(z) > 0$ then, $L(z) = \max_{p,q \in Q} \left\{ i(p) \cdot \mu^*(p, z, q) \cdot f(q) \right\}$ (3.5)

Since Q is finite and $z = x_1 x_2 \cdots\cdots x_m$ is of finite length, $L(z)$ is finite .Now consider any term in $L(z)$,let it be $w(z)$. Then,

$$\begin{aligned} w(z) = i\ (p_1) \cdot \mu(p_1, a_{11}, p_{11}) \cdot \mu(p_{11}, a_{12}, p_{12}) \cdots\cdots \mu(p_{1n_1-1}, a_{1n_1}, p_2) \cdot \mu(p_2, a_{21}, p_{21}) \cdot \\ \mu(p_{21}, a_{22}, p_{22}) \cdots \mu(p_{2n_2-1}, a_{2n_2}, p_3) \cdots \mu(p_m, a_m, p_{m1}) \cdot \mu(p_{m1}, a_{m2}, p_{m2}) \cdots \\ \mu(p_{mn_m-1}, a_{mn_m}, p_{m+1}) \cdot f(p_{m+1}) \end{aligned} \quad (3.6)$$

where $p_i, p_{ij}, p_{m+1} \in Q$, $i = 1,2,3.....m$, $j = 1,2,3.....(n_i - 1)$, $x_i = a_{i1} a_{i2} ....a_{in_i} \in A$

and $z = x_1 x_2 ......x_m$

From the definition of M, we have

(i) $\mu(p_i, a_{ij}, p_{ij}) = \mu_1(p_i, a_{ij}, p_{ij})$ *where* $i = 1,2,....m,\ j = 1,2....n_i - 2,$

(ii) There exist $p_{in_i} \in Q_1,\ f_1(p_{in_i}) > 0,\quad i_1(p_{i+1}) > 0$ such that

$$\mu(p_{in_i-1}, a_{in_i}, p_{i+1}) = \mu_1(p_{in_i-1}, a_{in_i}, p_{in_i}) \cdot f_1(p_{in_i}) \cdot i_1(p_{i+1}), \qquad i = 1,2....(m-1).$$

$$w(z) = i_1(p_1).\mu_1(p_1, a_{11}, p_{11}) \cdot \mu_1(p_{11}, a_{12}, p_{12}) \cdots\cdots \mu_1(p_{1n_1-1}, a_{1n_1}, p_{1n_1}) \cdot f_1(p_{1n_1}) \cdot i_1(p_2) \cdot$$
$$\mu_1(p_2, a_{21}, p_{21}) \cdot \mu_1(p_{21}, a_{22}, p_{22}) \cdots \mu_1(p_{2n_2-1}, a_{2n_2}, p_{2n_2}) \cdot f_1(p_{2n_2}) \cdots\cdots$$
$$i_1(p_m) \cdot \mu_1(p_m, a_{m1}, p_{m1}) \cdots \mu_1(p_{mn_m-1}, a_{mn_m}, p_{m+1}) \cdot f_1(p_{m+1})$$

$$w(x_1 x_2 \cdots\cdots x_m) = \left(i_1(p_1) \cdot \mu_1^*(p_1, x_1, p_{1n_1}) \cdot f_1(p_{1n_1})\right) \cdot \left(i_1(p_2) \cdot \mu_1^*(p_2, x_2, p_{2n_2}) \cdot f_1(p_{2n_2})\right) \cdots$$
$$\left(i_1(p_m) \cdot \mu_1^*(p_m, x_m, p_{m+1}) \cdot f_1(p_{m+1})\right)$$
$$= w_1(x_1) \cdot w_1(x_2) \cdot w_1(x_3) \cdots w_1(x_m)$$

This is true for every term in $L(z)$,

$$\max_{p,q \in Q}\left\{i(p) \cdot \mu^*(p, z, q) \cdot f(q)\right\} = \max_{p_1, p_{1n_1} \in Q_1}\left\{i_1(p_1) \cdot \mu_1^*(p_1, x_1, p_{1n_1}) \cdot f_1(p_{1n_1})\right\} \cdot$$
$$\max_{p_2, p_{2n_2} \in Q_1}\left\{i_1(p_2) \cdot \mu_1^*(p_2, x_2, p_{2n_2}) \cdot f_1(p_{2n_2})\right\} \cdots \max_{p_m, p_{m+1} \in Q_1}\left\{i_1(p_m) \cdot \mu_1^*(p_m, x_m, p_{m+1}) \cdot f_1(p_{m+1})\right\}$$
$$i.e., L(z) = L_1(x_1) \cdot L_1(x_2) \cdots L_1(x_m) \qquad x_i \in A, \qquad i = 1, 2, ...m$$

Similarly, we have i.e., $L_1(x_1) \cdot L_1(x_2) \cdots L_1(x_m) = L(z)$.

4. CONCLUSION

In this paper, the authors have made an attempt to study complete and some closure properties such as union, intersection, concatenation, Kleene's closure on an mwfa. We have made a humble beginning in this direction, however, many concepts are yet to be changed into weighted automata in the context of mwfa.

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