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Comparative Study of Homotopy Perturbation Method and Genocchi Polynomial Method for First Order Fractional Differential Equation

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ABSTRACT

In this paper, we consider fractional differential equations of order $0 < \sim \leq 1$ with initial condition $y(a) = b$. Here, we propose a technique to find the numerical solutions of first order fractional differential equations by using the Genocchi polynomials. Also, we solve the equations by Homotopy perturbation method and compare the obtained results with present Genocchi polynomial method. To illustrate the applications of our results we solve some numerical examples.

MSC: 65L60, 26A33, 34A08.

Keywords: Genocchi polynomial; Homotopy perturbation method, collocation method.

1. INTRODUCTION

In recent years, fractional calculus has penetrated in all branches of sciences and engineering where scientists are able to give a generalized flavor to all popular scientific models of fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc. (Cole, 1993; Debnath, L., 2003; Machado, J. A. T., 2010; El-Sayed, A., 2007; Machado, J.A.T., Galhano A. M. S. F., 2012; Das, S. Gupta, P., 2001) and as a result, fractional differential equations have become an important tool in mathematical modeling. There are many possible generalizations of $\frac{d^n}{dx^n}\left(f(x)\right)$, viz. in terms of Riemann-Liouville

(Samko *et al*., 1987; Podlubny, 2002), Caputo (Caputo, 1967; Diethelm, 2010), Weyl (Samko *et al*., 1987; Podlubny, 2002), Jumarie (Jumarie, 2005; Jumarie, 2006), Hadamard (Samko *et. al*., 1987; Podlubny, 2002), Davison and Essex (Davison, M., Essex, C., 1998), Riesz (Samko, S. G *et al*., 1987; Podlubny, 2002), Erdelyi-Kober (Samko *et al*., 1987; Podlubny, 2002), and Coimbra (Coimbra, 2003). Although theoretical concepts of fractional differential equations(FDEs) (Ahmed *et al*., 2007; Kilbas *et al*., 2006) are established well, the difficulties arise while handling the real world problems. In order to overcome these difficulties, scientists and mathematicians are putting continuous effort to make the existing numerical techniques for ODEs and PDEs compatible to FDEs. Some of numerical solution techniques for this kind of equations were studied in earlier works. In (Demirci, Ozalp, 2012) Demirci and Ozalp have introduced a technique to solve FDEs by using the exact solutions of corresponding integer order differential equation.

Many FDEs of physical importance are solved by operational matrix method (Saadatmandia and Dehghan, 2010), Laplace transform method (Kexue and Jigen., 2011), Adomian decomposition method (Hua et. al. 2008; Ray, Bera, 2005), Homotopy perturbation method (Chowdhury *et al*., 2007; He, 1999; He, 2000; Abdulaziz *et al*., 2008), nonstandard finite difference methods (Moaddya et. al., 2011), variational iteration method (Khan et. al., 2011), differential transform method (Erturk and Momani, 2008), tanh function method (Yusuf and Duzgun, 2018), predictor-corrector method (Diethelm *et al*., 2002), G'/G method (Baishya and Rangarajan, 2018).

In the present study, we consider linear and nonlinear fractional differential equations of order $0 < \alpha \leq 1$, which are used in modelling some equations (Kilbas *et al*., 2006; Glockle and Nonnenmache, 1995; Metzle, 1995). We use Genocchi polynomials (Araci, 2012; Bayad and Kim, 2010) with collocation method to solve first FDEs and obtain their numerical solution in terms of a series. We refer this as Genocchi Polynomial Method(GPM). Also the given FDEs are solved by Homotopy perturbation method (HPM) and compared the results with GPM. The comparison of exact solutions, the solution obtained by HPM and GPM is presented in Tables 1 and 2.

This paper is organized as follows: Section 2 is devoted to properties of Genocchi polynomials and basic definitions. In section 3 GPM and HPM are presented. Section 4 is concerned with the numerical experiment, results and error analysis of the illustrative problems, Finally, conclusion of the proposed work is discussed in section 5.

2. PRELIMINARY DEFINITIONS

Definition 1: Gamma Function

The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers. Its definition is given by,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\ dt, x \in R^+$$

The Gamma function has some unique properties. By using its recursion relations we can obtain formulas

$$\Gamma(x+1) = x\Gamma(x)\,, \quad x \in R^+$$
$$\Gamma(x) = (x-1)!\,, \quad x \in N$$

Definition 2: The Beta function

The Beta function is defined by a definite integral. Its definition is given by,

$$\beta(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}dt\,, x,y \in R^+$$

The Beta function can also be defined in terms of the Gamma function is,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Definition 3: The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t- \ )^{\alpha-1} f(\tau)d\tau(\alpha > 0)$$
$$J^0 f(t) = f(t)$$

Properties of Riemann-Liouville fractional integrals as follows,

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$

2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t),$

3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

Definition 4: The fractional derivative $(D)^\alpha$ of $f(t)$ in the Caputo's sense is defined as
$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t- \ )^{n-\alpha-1} f^{(m)}(\tau)d\tau \; for \;\; m-1 < \alpha \leq m, m \in N, t > 0, f \in C_{-1}^m.$

Definition 5: Genocchi polynomials

Genocchi numbers and polynomials have been extensively studied in many branches of mathematics such as elementary number theory, complex analytic number theory, homotopy theory, differential topology (differential structures on spheres) and quantum physics (quantum groups). The Genocchi numbers G_n and polynomials $G_n(x)$ are usually defined respectively by means of the exponential generating functions (Araci , 2012, 2014; Bayad and Kim, 2010):

$$Q(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!},\ (|t|) < ®$$

$$Q(t,x) = \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!},\ (|t|) < ®$$

Where $G_n(x)$ is the Genocchi polynomial of degree n and is given by

$$G_n(x) = \sum_{k=0}^{\infty}\binom{n}{k}G_{n-k}x^k$$

G_{n-k} here is the Genocchi number which can also be obtained from

$$G_n = 2\left(1-2^n\right)B_n \qquad \text{(a)}$$

Where B_n is the well-known Bernoulli number.

The first few Genocchi polynomials are:

$G_0(x)=0,\quad G_1(x)=1,\; G_2(x)=2x-1,\; G_3(x)=3x^2-3x,\quad G_4(x)=4x^3-6x^2+1.$

Some of the important properties of Genocchi polynomial are:

I. $\int_0^1 G_n(x)G_m(x) = \dfrac{2(-1)^n n!m!}{(m+n)!}G_{m+n},\; n,m \geq 1$

II. $\dfrac{d}{dx}\left(G_n(x)\right) = nG_{n-1}(x), n \geq 1$ (b)

III. $G_n(1)+G_n(0)=0, n>1$

3. DESCRIPTION OF GENOCCHI POLYNOMIAL METHOD

Consider the general fractional first order differential equation,

$$D^{\sim} y(x) = f(x,y) \qquad (1)$$

With initial condition $y(a)=b$, where $0<\sim\leq 1$, a and b are real constants.

Assume that, $y(x)=\sum a_i G_i(x)$ (2)

Truncate the equation (2) as, $y(x)\approx \sum_{i=0}^{m} a_i G_i(x)$ (3)

where $G_i(x)$ are Genocchi polynomials and m is any positive integer,

Now, Substituting equation (3) in (1) we get,

$$D^{\sim}\left(\sum_{i=0}^{n} a_i G_i(x_j)\right) = f\left(x,\sum_{i=0}^{n} a_i G_i(x_j)\right) \qquad (4)$$

For the fractional values of $\sim$ we apply Caputo derivative as defined in section 2. Applying the collocation points $\langle x_j \rangle = \{y(\frac{j}{m})\},\ \forall\ j=1,2,3,...,m-1,$ where m is any positive integer, in equation (4), we obtain $m-1$ equations and along with given initial equation we have a system of m algebraic equations. By solving this system using suitable solver, we obtain the values of m unknown coefficients and hence the solution of the FDE.

4. NUMERICAL EXAMPLES

Example 1. Consider a fractional order differential equation (Akrami and Hussien, 2013),

$$D^{\sim} y(x) + y(x) = 0, \quad 0 < \sim \leq 1 \tag{5}$$

With initial condition, $y(0) = 1$ and the exact solution is $y(x) = e^{-x}$. Here, we solved the Example1 by both GPM and HPM. Fig. 1 represents numerical solutions of example 1 by GPM at different values of $\sim$. Fig. 2 shows graphical comparison of GPM solution to exact solution. Table 1 represents absolute error(AE) of GPM and HPM.

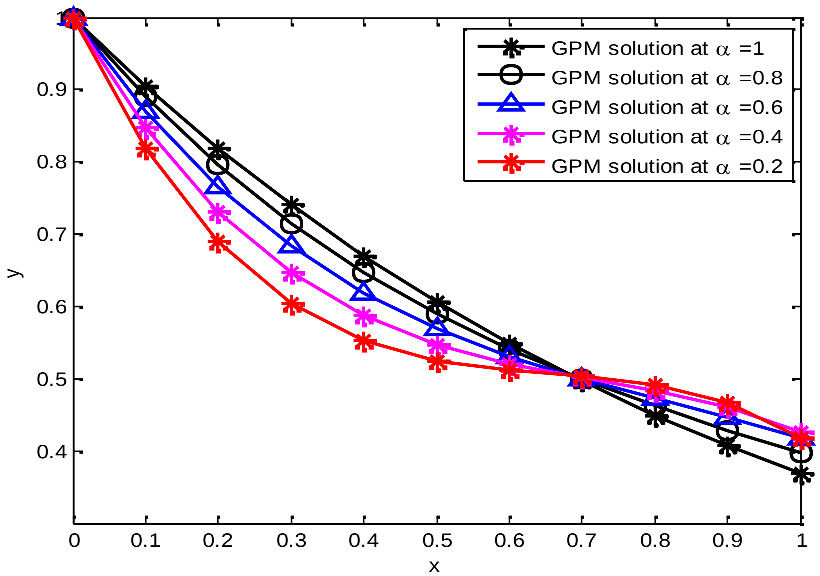

Fig. 1. Graphical representation for the example 1 at different values of $\sim$.

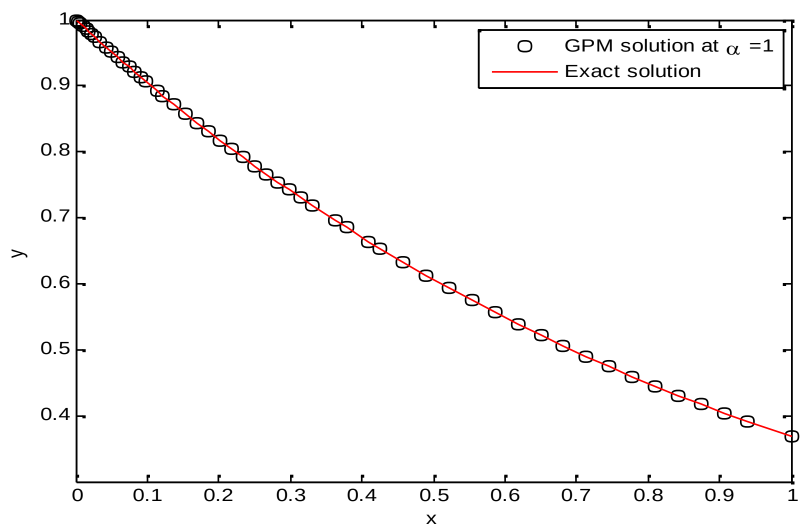

Fig. 2. Graphical comparison of exact solution with GPM solution.

Table. 1: Absolute error (AE) of GPM and HPM.

x	Exact solution	AE by HPM at n=5	AE by HPM at n=7	AE by GPM at m=5	AE by GPM at m=10
0.1	0.904837	2.03596×10^{-9}	1.03596×10^{-9}	5.4219×10^{-5}	3.6859×10^{-14}
0.2	0.818731	8.7078×10^{-8}	1.07798×10^{-9}	5.9004×10^{-5}	6.9367×10^{-13}
0.3	0.740818	9.70682×10^{-7}	1.68172×10^{-9}	5.0418×10^{-5}	2.5757×10^{-14}
0.4	0.670320	5.38004×10^{-6}	1.70356×10^{-8}	4.3679×10^{-5}	9.7145×10^{-14}
0.5	0.606531	2.02437×10^{-5}	1.52713×10^{-7}	4.0964×10^{-5}	6.8789×10^{-13}
0.6	0.548812	5.96361×10^{-5}	3.91094×10^{-7}	3.8470×10^{-5}	8.8707×10^{-14}
0.7	0.496585	1.48388×10^{-4}	1.32679×10^{-6}	3.2805×10^{-5}	6.7168×10^{-14}
0.8	0.449329	3.26298×10^{-4}	3.82012×10^{-6}	2.6769×10^{-5}	2.3433×10^{-12}
0.9	0.406567	6.52909×10^{-4}	9.69774×10^{-6}	3.4585×10^{-5}	3.4314×10^{-12}
1	0.367879	1.21278×10^{-3}	2.2299×10^{-5}	8.6633×10^{-5}	2.9524×10^{-12}

Example 2. Consider a fractional order differential equation (Khodabakhshi *et al*., 2014),

$$D^{\sim} y(x) + y^2(x) = 1 \tag{6}$$

With initial condition, $y(0) = 0$. The exact solution is $y(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ at $\sim = 1$.

Here, we solved the Example 2 by both GPM and HPM. Fig. 3 and Fig.4 represents respectively numerical solutions by GPM at different values of $\sim$ and graphical comparison of GPM solution to exact solution. Table 2 represents comparative analysis of AE of GPM and HPM.

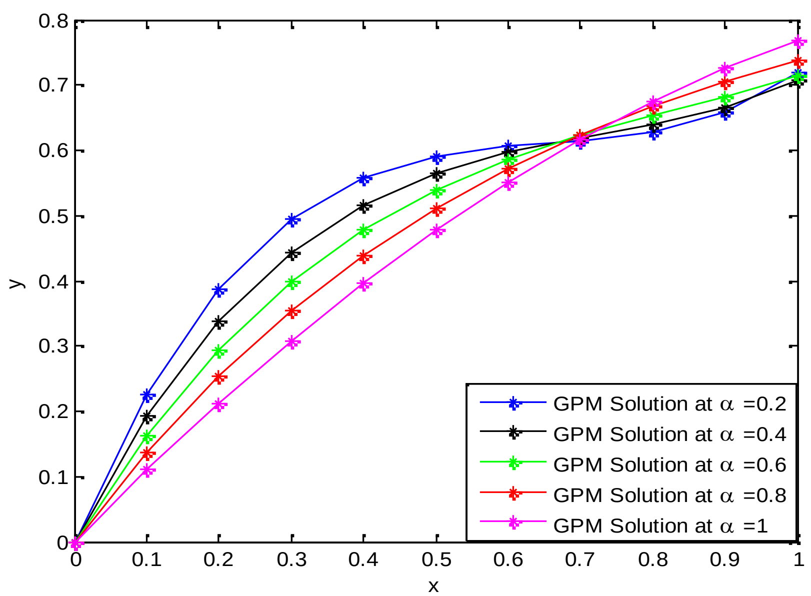

Fig. 3. Graphical representation for the example 2 at different values of $\sim$.

Table. 2: Absolute error (AE) of GPM and HPM with exact solution.

X	Exact solution	AE by HPM at n=5	AE by HPM at n=7	AE by GPM at m=10
0.1	0.099668	8.82489×10^{-14}	2.77556×10^{-17}	5.64357×10^{-7}
0.2	0.197375	1.78623×10^{-10}	4.69347×10^{-14}	5.23139×10^{-7}
0.3	0.291313	1.51484×10^{-8}	2.01544×10^{-11}	5.05358×10^{-7}
0.4	0.379949	3.49113×10^{-7}	1.468×10^{-9}	4.68336×10^{-7}
0.5	0.462117	3.9296×10^{-6}	4.03411×10^{-8}	4.33734×10^{-7}
0.6	0.53705	2.80613×10^{-5}	5.97356×10^{-7}	3.91053×10^{-7}
0.7	0.604368	1.46218×10^{-4}	5.7665×10^{-6}	3.41767×10^{-7}
0.8	0.664037	6.0454×10^{-4}	4.06728×10^{-5}	3.30136×10^{-7}
0.9	0.716298	2.09397×10^{-3}	2.25663×10^{-4}	2.63472×10^{-7}
1	0.761594	6.30708×10^{-3}	1.03597×10^{-3}	2.00704×10^{-8}

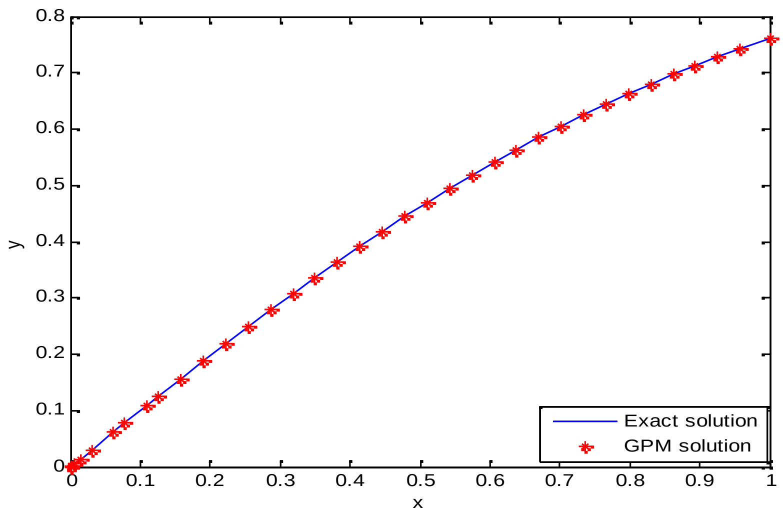

Fig. 4. Graphical comparison of exact solution to GPM solution.

5. CONCLUSION

In this paper, we have developed Genocchi polynomial collocation method for solutions of first odrer FDEs with initial condition. Numerical solutions obtained by using the presented method are compared with the corresponding exact solutions and solutions obtained by using HPM. The comparison of the results are shown in Table 1 and 2. It is observed that, in case of linear FDEss Genocchi polynomial collocation method performs very well. Even though in case of nonlinear FDEs, this method's performance is little lower than HPM, it will work as a strong tool to solve nonlinear FDEs when singularity is present. Because in such cases computation with HPM becomes cumbersome.

Conflict of Interest

I confirm that this article contents have no conflict of interest.

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REFERENCES

1. Abbas Saadatmandia, Mehdi Dehghan, A new operational matrix for solving fractional-order differential equations, *Computers and Mathematics with Applications*, 59(3), 1326-1336 (2010).
2. Abdulaziz, O., Hashim, I., Momani, S., Application of Homotopy-perturbation fractional IVPs, *Journal of Computational and Applied Mathematics*, 216(2), 574–584 (2008).
3. Ahmed, E., El-Sayed, A.M.A., and El-Saka, H.A.A., Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies model, *Journal of Mathematical Analysis and Applications*, 325(1), 542–553 (2007).
4. Akrami, M. H., and Hussien Erjaee, G., The operational matrix of fractional integration for shifted Legendre polynomials. *Iranian Journal of Science and Technology Transaction A Science*, 26(1), 25-31 (2013).
5. Araci, S., Novel identities for q-Genocchi numbers and polynomials. *Journal of Function Space and Applications*, Article ID 214961, 13 (2012).
6. Bayad, A., Kim, T., Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials. *Advanced Studies in Contemporary Mathematics*, 20, 247-253 (2010).
7. Caputo, M., Linear models of dissipation whose Q is almost frequency independent, part II, *Geophysical Journal International*, 13(5), pp. 529–539 (1967).
8. Chandrali Baishya and Rangarajan, R., A New Application of G'/G-Expansion Method for Travelling Wave Solutions of Fractional PDEs, *International Journal of Applied Engineering Research*, 13(11), 9936-9942 (2018).
9. Chowdhury, M. S. H., Hashim, I. & Abdulaziz, O. Application of Homotopy perturbation Method to Nonlinear Population Dynamics Models. *Physics Letter*, 368, 251-228 (2007).
10. Coimbra, C. F. M., Mechanics with Variable-Order Differential Operators, *Annalen der Physik*, 12(11-12), 692–703 (2003).
11. Cole, K.S., Electric Conductance of Biological Systems, *Proceedings of Cold Spring Harbor Symposium on Quantitative Biology*, 12, 107–116 (1993).
12. Das. S., Gupta, P., A Mathematical Model on Fractional Lotka-Voltera Equation, *Journal of Theoretical Biology*, 227(1), 1-6 (2001).
13. Davison, M., and Essex, C., Fractional Differential Equations and Initial Value Problems, *The Mathematical Scientist*, 23(2), 108–116 (1998).
14. Debnath, L.,Recent Applications of Fractional Calculus to Science and Engineering, *International Journal Mathematics and Mathematical Science*, 2003(54), 3413-3442 (2003).
15. Diethelm, K., Ford, N. J., and Freed, A. D., A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations, *Nonlinear Dynamics*, 29(1-4), 3-22 (2002).

16. Diethelm, K., The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type, Springer-Verlag, Berlin Heidelberg, 1617-9692 (2010).
17. Elif Demirci, Nuri Ozalp, A Method for Solving Differential Equations of Fractional Order, *Journal of Computational and Applied Mathematics*, 236(11), 2754–2762 (2012).
18. El-Sayed A., El-Mesiry A., El-Saka H., On the fractional order logistic equation. *Applied Mathematics Letter*, 20(7), 817–823 (2007).
19. Ertürk, V. S., Momani, S., Solving Systems Of Fractional Differential Equations Using Differential Transform Method, *Journal of Computational and Applied Mathematics*, 215, 142 – 151 (2008).
20. Glöckle, W., Nonnenmacher, T. F., A Fractional Calculus Approach to Self-Similar Protein Dynamics. *Biophysical Journal*, 68 (1), 46–53 (1995).
21. He, J. H. Homotopy Perturbation Technique. *Computational Methods for Applied Mechanics and Engineering*, 178(3/4), 257-262 (1999).
22. He, J. H. A Coupling Method of Homotopy Technique and Perturbation Technique for Nonlinear Problems. *International Journal of Non-Linear Mechanics*, 35, 37-43 (2000).
23. Hua, Y., Luoa Z. Lu, Y., Analytical Solution of the Linear Fractional Differential Equation by Adomian Decomposition Method. *Journal of Computational and Applied Mathematics*, 215, 220 – 229 (2008).
24. Jumarie, G., On the Solution of the Stochastic Differential Equation of Exponential Growth Driven by Fractional Brownian Motion. *Applied Mathematics Letters*, 18(7), 817–826 (2005).
25. Jumarie, G., Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. *Computers & Mathematics with Applications*, 51 (9-10), 1367–1376 (2006).
26. Kilbas, A. A., Srivastava, H.M., Trujillo, J.J., Theory and Applications of Fractional Differential Equations, 204, Elsevier (2006).
27. Li Kexue, Peng Jigen, Laplace Transform and Fractional Differential Equations. *Applied Mathematics Letters*, 24, 2019–2023 (2011).
28. Machado J.A.T., Galhano A.M.S.F., Fractional Order Inductive Phenomena Based on the Skin Effect. *Nonlinear Dynamics*, 68(1–2), 107–115 (2012).
29. Machado, J.A.T., Entropy Analysis of Integer and Fractional Dynamical Systems. *Nonlinear Dynamics*. 62 (1–2), 371–378 (2010).
30. Metzler, R., Schick, W., Kilian, H. G., and Nonnenmacher, T. F., Relaxation in Filled Polymers: A Fractional Calculus Approach. *The Journal of Chemical Physics*, 103 (16), 7180–7186 (1995).
31. Moaddya, K., Hashima, I., and Momanib, S., Non-Standard Finite Difference Schemes for Solving Fractional-Order Rössler Chaotic And Hyper Chaotic Systems, *Computers and Mathematics with Applications*, 62, 1068-1074 (2011).
32. Neda Khodabakhshi, S. Mansour Vaezpour & Dumitru Baleanu. Numerical Solutions of the Initial Value Problem for Fractional Differential Equations by Modification of the

Adomain Decomposition Method. *Fractional Calculus and Applied Analysis*, 17(2), DOI: 10.2478/s13540-014-0176-2 (2014).

33. Podlubny, I., Geometric and physical interpretation of fractional integration and fractional differentiation. *Fractional Calculus and Applied Analysis*, 5(4), 367–386 (2002).
34. Ray, S. S., Bera, R.K., An Approximate Solution of a Nonlinear Fractional Differential Equation by Adomian Decomposition Method, *Applied Mathematics and Computation*, 167, 561–571 (2005).
35. Samko, S. G., Kilbas, A. A., and Maritchev, O. I., Integrals and Derivatives of the Fractional Order and Some of Their Applications, in Russian, Naukai Tekhnika, Minsk, Belarus (1987).
36. Yasir Khan, Naeem Faraz, Ahmet Yildirim, Qingbiao Wu, Fractional Variational Iteration Method for Fractional Initial-Boundary Value Problems Arising in the Application of Nonlinear Science, *Computer and Mathematics with Applications*, 62(5), 2273-2278 (2011).
37. Yusuf Pandir and Hasan Huseyin Duzgun, Analytical Approach for the Fractional Differential Equations by Using the Extended tanh Method, *Wave in Random and Complex media*, 28(3), 399-410 (2018).