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Basic Concepts and Definitions of Compact Groups and Fourier Analysis

Birendra Kumar[1] and Bijay Kumar Singh[2]

[1]Research Scholar,
Department of Mathematics, VKSU. Ara., INDIA.
[2]Associate Professor and H.O.D.,
P.G. Department of Mathematics, H.D. Jain College, Ara., INDIA.

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ABSTRACT

This paper contains all basic concepts and definitions of compact Groups, Locally Compact Groups and Fourier Series.

Keywords: Topological Spaces, Metric Space**,** Banach Space, Hilbert Space, Locally Compact Space, Fourier Series.

1. INTRODUCTION

Here we have collected the basic concepts and definitions that can be used for further study.

1.1 Topology and topological spaces

Let X be a non-empty set. Consider the family J of subset of X satisfying the following conditions

(i) $\Phi \in J$, $X \in J$.

(ii) Arbitrary union of members of J is again a member of J.

i.e. if $\{G_i\}$ $i \in I$ be any arbitrary family of members of J, then $\underset{i \in 1}{\cup G i} \in J$

(iii) Intersection of any two members of J is again a member of J.

i.e. if G_1 and $G_2 \in J$ be any two members the $G_1 \cap G_2 \in J$.

In this case, the family J is called topology on X and the pair (X, J) is called topological space.

1.2 Open Set

Let (X, J) be any topological space where J is a topology on X, then the members of j are called open sets in X.

Remarks

In terms of open set

(i) Φ and X are open sets in X.
(ii) Arbitrary union of open sets in X is open.
(iii) Intersection of any two open seats in X is open.

1.3 Closed Set

Let (X, J) be any topological space where J is a topology on X. then a subset A of X is called closed in X if its complement A' ∈ J.

Hence A is closed in X ⟹ A' ∈ J.

Thus the complement of the members of topology J is closed in X.

1.4 Definition (Metric Space)

Let X be any non-empty set and consider a mapping d: X ⊗ X → R which satisfying the following conditions

(i) d(x,y) ≥ 0 ∀ x, y ∈ X.
(ii) d(x,y) = 0 ⇔ x = y.
(iii) d(x,y) = d (y,x) ∀ x,y, ∈ X.
(iv) d(x,y) ≤ $d\,(x,z) + d\,(z,y)$ ∀ x, y, z ∈ X.

In this case, 'd' is said to be metric (distance function) on the se X and the pair (X, d) is called metric space.

1.5 Definition (Hausdroff space/T_2-Space)

Let (X,J) be any top-space where J is a topology on X then X is said to be Hausdroff space or T_2 – space if for each pair, x, y ∈ X of distinct points. There exists open set G&H s.t. x ∈ G, y ∈ H and G ⋂ H = Φ.

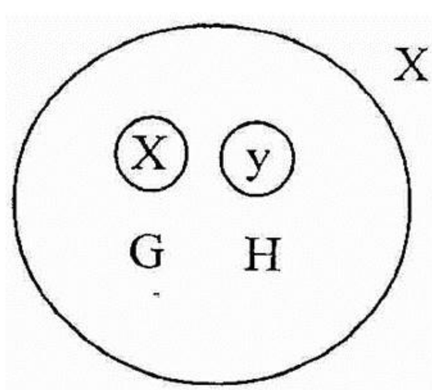

Figure: 1

1.6 Definition (Continuity in a Topological Space)

Let (X, J) and (Y, J') be any two topological spaces then a mapping $f : X \rightarrow Y$ is said to be a continuous at a point $x \in X$ if for open set V containing f(x) there exists a open set U containing x in X s.t. $\boxed{f(U) \subseteq V}$. Moreover f is said to be continuous in x if f is continuous at every point of X.

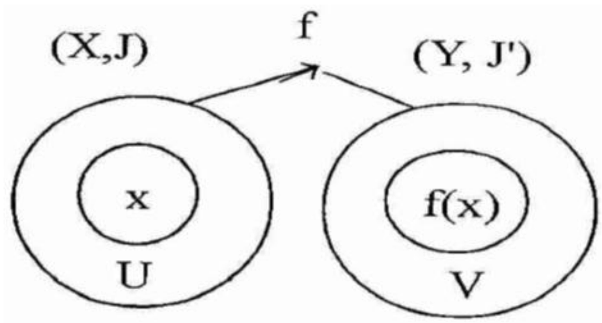

Figure: 2

1.7 Definition (Normed Linear Space)

Let X be any vector space (Linear space), then X is said to be normed linear space (denoted as ‖ ‖) if it satisfying the following conditions

(i) $\| x \| \geq 0 \ \forall x \in X$.

(ii) $\| x \| = 0 \Longrightarrow x = 0$.

(iii) $\| \alpha x\| = | \alpha | \ \| x \| \ \forall x \in X$ and $\alpha \in F$.

(iv) $\| x + y \| \leq \| x \| + \| y \|$ (Triangle inequality)

In this case the pair (X, ‖ ‖) is called nermed linear space.
Note: If the norm satisfying the conditions (iii) and (iv) then it is called semi-norm.

1.8 Definition (Convergent sequence in a normed linear space)

Let (X, ‖ ‖) be a normed linear space, then a sequence $<x_n>$ in X is said to be converge to an element $x_0 \in X$, if for given $\in >0$, there exists a positive integer m_0 such that $\| x_0 - x_n \| < \in \forall n \geq m_0$

If x_n converges to x_0 we can write $\lim_{n \to \infty} x_n = x_0$ or $x_n \rightarrow x_o$ as $n \rightarrow \infty$.

Hence if $x_n \rightarrow x_0$ iff $\| x_n - x_0 \| \rightarrow 0$.

1.9 Definition (Cauchy sequence in Normed linear space)

Let (X, ‖ ‖) be a normed linear space, then a sequence $<x_n>$ in X is called Cauchy sequence if for a given $\in >0$, there exists a positive integer m_0 such that $\| x_m - x_n \| < \in \forall m, n \geq m_0$

1.10 Definition (Complete Metric space)

Let (X, ‖ ‖) be a normed linear space, then X is called complete metric space if every Cauchy sequence in X is convergent in X. In other words,

A metric space (x, d) in which every Cauchy sequence is convergent is said to be complete metric space.

1.11 Definition (Banach space)

A normed linear space which is complete as a metric space is called Banach space.

OR

A normed linear space in which every Cauchy sequence is convergent is called Banach space.

1.12 Definition (Hilbert Space)

A complex Banach space H is called Hilbert space if a complex number (x, y) called the inner product of x and y is satisfying the following properties: -

(i) $(\overline{x, y}) = (y,x)$

(ii) $(\alpha x + \beta y,z) = \alpha (x,z) + \beta (y,z)$

(iii) $(x,x) = \| x \|^2 \forall x, y, z \in H$ and $\alpha, \beta \in F$.

1.13 Definition (Connected space)

Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open sub-sets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

OR

A topological space X is said to be connected iff the only subsets of X that are both open and closed in X are the empty sat and X itself.

1.14 Definition (Covering of Top space)

A collection A of subsets of space X is said to cover X, or to be covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

1.15 Definition (Compact space)

A space X is said to be compact if every open covering A of X contains a finite sub collection that also covers X.

1.16 Definition (Locally compact space)

A topological space X is said to be locally compact if for each p ∈ X, there exists an open nhood N(p) such that $\overline{N(p)}$ is compact. Obviously, $\overline{N(p)}$ is also a nhood of P.

The continuous image of locally compact space may not be locally compact. However, if the map is open also, then the image of a locally compact space is always locally compact.

1.17 Definition (Product Space)

Let I be the index set. Let $\{x_i\}$, $i \in I$ be a family of topological space. Let

$X = \Pi X_i$ then $_{X : I} \rightarrow \cup X_{i.}$

$i \in I$ $\quad\quad\quad\quad$ $i \in I$

Such that $x_i \in X_i$ for each $i \in I$.

Let the function $f_i : X \rightarrow X_i$ be defined by $f_i (x) = x_i$ for each $x \in X$. Let J be the weakest (x, J) topology on X in which each f_i is continuous, then is a topological space. This space is called the product space of the space X_i. The topology J on X is called the product topology.

1.18 Definition (Connected Space)

A topological space X is said to be connected if there exists no non-empty open sets A and B such that AUB = X.

A subset of a topological space is said to be connected if it is a connected space in the relative topology.

1.19 Definition (Locally Connected Space)

A topological space X is said to be locally connected if each open nhood of every point of X contains a connected open nhood.

1.20 Definition (topological Groups)

A topological space G is called a topological group if

(i) G is a group with respect to a binary operation.

(ii) The mapping $g_1 : G X G \rightarrow G$ defined by $g_1 (x,y) = xy$ from G X G onto G is continuous in both the variables simultaneously and

(iii) The mapping $g_2 : G \rightarrow G$ defined by $g_2 (x) = x^{-1}$ from G onto G is also continuous. The mapping g_1 and g_2 are called the multiplication map and inversion mapping respectively.

Examples: (1) R, the set of real numbers is a topological goup with respect to the binary operation addition '+' with the usual metric topology.

(2) R\{o}, the set of all non-zero real numbers with the usual topology is a topological group with respect to the ordinary multiplication 'o'.

(3) A group with respect to the discrete topology is a topological group.

1.21 Definition (Separation in topological groups)

Let G be a topological group. Then the following statements are equivalent:

(i) G is T_0 - space

(ii) G is a T_1 – space

(iii) G is a T_2 - space

(iv) Let T_0 – space, G be a topological group then G is completely regular and hence regular.

1.22 Definition (Subgroup of a topological group)

Let H be a subgroup of a topological group G, then H is called a topological subgroup or simply a subgroup of topological group G.

If e is the identity of a topological group G, then $\{\bar{e}\}$ in G is a closed invariant subgroup of G and is the smallest closed subgroup of G.

1.23 Definition (Locally compact topological group)

A topological group G is called locally compact if and only if , there exists a compact nhood of the identity element e ∈ G.

Note:

Let H be a subgroup of a topological group G, then the following are obvious:-

(i) If H is locally compact, then H is closed.

(ii)If G is locally compact, then $\frac{G}{H}$ is locally compact.

(iii) A locally compact topological group G has no proper sub group if and only if for each nhood V of the identity e,

G = $\bigcup_{n=1}^{\infty} V^n$.

1.24 Definition (Riemann Integration)

Let f : [a, b] → R be a bounded function. The real numbers

$$\int_{\underline{a}}^{b} f(x)dx = \text{Sup } \{L\ (P, f)\ |P \text{ a partition of f}\}$$

and $$\int_{a}^{\overline{b}} f(x)dx = \text{inf } \{U\ (P, f)\ |P \text{ a partition of f}\}$$

are called the lower integral and the upper integral of f respectively. the function of f is said to be Riemann integrable on [a, b] if

$$\int_{a}^{b} f(x)dx = \int_{\underline{a}}^{b} f(x)dx = \int_{a}^{\overline{b}} f(x)dx$$

1.25 Definition (Periodic function)

A function is f(x) said to be periodic with period of P if for all x, f(x+P)= f(x), where P is a positive constant. The least value of P > 0 is called the least period or simply the period of f(x).

Example: The function sin x has periods $2\pi, 4\pi, 6\pi \ldots\ldots$. Since sin(x+2π), sin(x+4π)……… all equal to sin x. However, 2π is the least period or period of sin x.

1.26 Definition (Fourier series)

Let f(x) be defined in the interval (-L, L) and determined outside of this interval by f(x + 2L) = f(x). i.e. assume that f(x) has the period 2L. the Fourier series or Fourier expansion corresponding to f(x) is defined to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \quad (1)$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} dx$$

Where n =0, 1, 2…. (2)

Definition (Odd and Even functions)

A function f(x) is called odd if f(-x) = -f(x).

Example: x^3. $x^5 - 3x^3 + 2x$, sin x, tan 3x are odd functions.

A function is called even if f(-x) = f(x) thus x^4, $2x^6$-$4x^2$+5, cos x are even functions.

In the Fourier series corresponding to an odd function, only sine terms can be present an even function, only cosine terms can be present.

1.27 Measure and measurable function

Let X be any non-empty set. Let S be a subset of X, then a real valued function f on X is called S-measurable if the set $\{x \in X : f(x) > \alpha\}$ is a member of S for every real α.

If a function on X is complex- valued, then f is called S-measurable provided its real and imaginary parts are S-measurable.

1.28 Borel Measurable

If X is a topological space and S is the σ – algebra of Borel set of X, a S-measurable function f on X is called Borel measurable.

1.29 Additive measure

A set function μ is called countably additive measure or simply a measure on S if it satisfying the following conditions

(i) $\mu(\Phi) = 0$

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in S$ such that $A \cap B = \Phi$, where μ is the finitely additive and

(iii) $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$

Whenever each $A_k \in S$, $\bigcup_{k=1}^{\infty} A_k \in S$ and the A_k's are pair wise disjoint, then v is called σ – additive (Countably additive or completely additive) on S.

1.30 Measure space

Let X be a set. Let μ be a countably additive measure on a σ – algebra. A of subset of X, then the triple (X, A, μ) is called a measure space. If $\mu(X) < \infty$, then (X, A, μ) is called a finite measure space with finite measure μ.

1.31 Complete measure space

Let μ be a measure defined on A such that all subsets of sets of measure zero are measurable then μ is called a complete measure and (X, A, μ) is called a complete measure space.

CONCLUSION

This paper is the collection of basic concepts and definitions that can be used for further studied.

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