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Common Fixed Point Theorems in Complex Valued Metric Spaces

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ABSTRACT

In the present paper we define the concept of metric space, b-metric space, complex b-metric space and proved fixed point results. Recently we established and proved several results on common fixed point in complex valued metric spaces.

Keywords: Metric space, b-metric space, complex valued-metric space, complex valued b- metric space.

INTRODUCTION

In 1998, Czerwik[15] introduced the concept of b-metric space. Recently, Rao *et al*.[16] developed the notion of complex valued b-metric spaces and proved fixed point results. Very recently Singh *et al*.[17] established and proved several results on common fixed point for a pair of mappings satisfying mare general contraction conditions portrayed expressions having point-dependent control functions as coefficients in complex valued metric spaces.

Moreover Sumit Chandok and Deepak Kumar[18] proved some common fixed pint theorems for four self maps having weakly compatibility in the concept of complex valued metric spaces.

Definition 1: Let X be any non-empty set and let d: $x \otimes x \rightarrow [0, \infty)$ be a function satisfying following conditions:

(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0 \Leftrightarrow x = y$
(iii) $d(x, y) = d(y, x)$

(ii) $d(x, y) = d(x, z) + d(z, y) \;\forall\; x, y, z \in X$.

If d is distance function on X. Then the pair (X, d) is called metric space.

Definition 2: A sequence $\{x_n\}$ in metric space (x, d) is called Cauchy sequence if for a given $\epsilon > 0$ there exists a number $n_0 \in N$ such that $\forall\; m, n > n_0$, $d(x_m, x_n) < \epsilon$.

Definition 3: A sequence $\{x_n\}$ in metric space (x, d) is convergent to $x \in X$ if $\underset{n \to \infty}{\text{lt}}\, x_n = x$. In this case x is called a limit of the sequence $\{x_n\}$.

Definition 4: A metric space (x, d) is called complete if every Cauchy sequence is convergent.

Definition 5: Let (X, d) be a metric space and $T : X \to X$ be a mapping then T is said to be expansive mapping if for every $x, y \in X$ there exists a number $r > 1$ such that

$D(T_x, T_y) \geq rd(x,y)$

Definition 6: Let X be a non empty set and $s \geq 1$ a given real number. A function $d: X \otimes X \to C$ is called a b-metric space on x if d satisfying the following conditions:

(i) $o \leq d(x, y)$ for all $x, y \in X$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iv) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

Then d is called a b-metric on X, and (X, d) is called a b-metric space. It is obvious that a b-metric space with base s=1 is a metric space.

Definition 7: Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(i) A sequence $\{x_n\}$ is called convergent if and only if $x \in X$ such that

$d(x_n, x) \to o$ when $n \to \infty$

(ii) $\{x_n\}$ is a Cauchy sequence if and only if

$d(x_n, x_m) \to o$ when $n, m \to \infty$

In general, a b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Definition 8: Let X be a non empty set. A mapping $d : X \otimes X \to C$ satisfies the following conditions:

(i) $o \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = o$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + (x, y)$ for all $x, y, z \in X$

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example: Let X = C. Define the mapping $d: X \otimes X \to C$ by $d(z_1, z_2) = e^{ik}\, | z_1 - z_2 |$, where $K \in R$. Then (X, d) is a complex valued metric space.

Definition 9: Let X be a non empty set and let $s \geq 1$ be a given real number. A function $d: X \otimes X \to C$ is called a complex valued b-metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) o ≤ d(x, y) and d(x, y) = o if and only if x = y
(ii) d (x, y) = d (y, x)
(iii) d (x, y) ≤ d [(x, z) + (x, y)]
The pair (X, d) is called a complex valued b-metric space.
Example: Let X = [o, 1]. Define a mapping d: X⊗X→C by d(x, y) = $|x-y|^2 + i\ |x-y|^2$ for all x, y ∈ X. then (X, d) is a complex valued b-metric space with s=2.

Definition 10: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ ba a sequence in X and x ∈ X.
(1) If for every c ∈C with 0 ≺ c, there exits n_0 ∈ N such that $d(x_n, x) \prec c$ for all n > n_0, then $\{x_n\}$ is said to be converges to x and x is a limit point of $\{x_n\}$. We denote this by $x_n \to x$ as n → ∞ or $\lim_{n\to\infty} x_n$ = x.
(2) If for every c ∈C with 0 ≺ c, there exits n_0 ∈ N such that for all n > n_0, $d(x_n, x_{n+m}) \prec c$ where m ∈ N, then $\{x_n\}$ is said to be Cauchy sequence.
(3) If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued b-metric space.

Lemma 1: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence inX. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as n → ∞.

Lemma 2: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence inX. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as n → ∞, where m ∈ N.

Definition 11: Let A and S be self mappings on a set X, if w = Ax =Sx for some x in X, then x is called coincidence point of A and S and w is called a point of coincidence of A and S.

Definition 12: A pair of self mappings A, S : X → X is called weakly compatible if A and S commute at their coincidence point. That is, if there be a point u such that Au = Su, then ASu – SAu, for each u ∈ X.

Definition 13: Let T, S : X → X be two self mappings of a complex valued metric space (X, d). The pair (T, S) is said to satisfy (E. A.) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$ = t, for some t ∈ X.

Theorem 1: Let (X, d) be a complete complex valued b-metric space with the coefficient s ≥ 1 and let T, S : X → X be self mappings satisfying:

$$D(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu\{d(x,Ty)d(y,Ty)+d(y,Sx)d(y,Ty)\}}{1+d(x,Ty)+d(y,Sx)} \quad (1)$$

for all x, y ∈ X, where λ, μ are nonnegative reals with s λ + μ < 1. Then the mapping S and T have unique common fixed point in X.

Proof: For any arbitrary point, x_0 in X. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1}, \text{ for } n = 0, 1, 2, \ldots \quad (2)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let x = x_{2n} and y = x_{2n+1} in (1), we have
$d(x_{2n+1}, x_{2n+2})$
$d(Sx_{2n}, Tx_{2n+1})$

$$\precsim \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu\begin{Bmatrix} d(x_{2n},Tx_{2n+1})\, d(x_{2n+1},Tx_{2n+1})+ \\ d(x_{2n+1},Sx_{2n})d(x_{2n+1},Tx_{2n+1}) \end{Bmatrix}}{1+d\,(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})}$$

$$= \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu\{d(x_{2n},x_{2n+2})\, d(x_{2n+1},x_{2n+2})\}}{1+d\,(x_{2n},x_{2n+2})},$$

which implies that

$$|d\,(x_{2n+1},x_{2n+2})|$$

$$\leq \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu\{|d(x_{2n},x_{2n+2})|\,|d(x_{2n+1},x_{2n+2})|\}}{|1+d\,(x_{2n},x_{2n+2})|}.$$

Since $|d\,(x_{2n},x_{2n+2})| < |1 + d\,(x_{2n},x_{2n+2})|$, we have

$$|d\,(x_{2n+1},x_{2n+2})| \leq \lambda\,|d\,(x_{2n},x_{2n+1})| + \mu\,|d\,(x_{2n+1},x_{2n+2})|,$$

and hence

$$|d\,(x_{2n+1},x_{2n+2})| \leq \frac{\lambda}{1-\mu}|d\,(x_{2n},x_{2n+1})|.$$

Similarly, we can show that

$$|d\,(x_{2n+2},x_{2n+3})| \leq \frac{\lambda}{1-\mu}|d\,(x_{2n+1},x_{2n+2})|.$$

Since $s\lambda + \mu < 1$ *and* $s \geq 1$, we get $\lambda + \mu < 1$. Therefore, with $\delta = \frac{\lambda}{1-\mu} < 1$, and for all $n \geq 0$, consequently, we get

$$\begin{aligned} |d\,(x_{2n+1},x_{2n+2})| &\leq \delta\,|d\,(x_{2n},x_{2n+1})| \\ &\leq \delta^2\,|d\,(x_{2n-1},x_{2n})| \\ &\leq \dots \leq \delta^{2n+1}\,|d\,(x_0,x_1)|. \end{aligned} \quad (3)$$

Thus, for $m > n$, $m, n \in N$, and since $s\delta = \frac{\lambda}{1-\mu} < 1$, we have

$$\begin{aligned} |d\,(x_n,x_m)| &\leq s\,|d\,(x_n,x_{n+1})| + s\,|d\,(x_{n+1},x_m)| \\ &\leq s\,|d\,(x_n,x_{n+1})| + s^2\,|d\,(x_{n+1},x_{n+2})| + s^2\,|d\,(x_{n+2},x_m)| \\ &\leq s\,|d(x_n,x_{n+1})| + s^2\,|d\,(x_{n+1},x_{n+2})| + s^3\,|d\,(x_{n+2},x_{n+3})| + s^2\,|d\,(x_{n+3},x_m)| \\ &\leq s\,|d(x_n,x_{n+1})| + s^2\,|d\,(x_{n+1},x_{n+2})| + s^3\,|d\,(x_{n+2},x_{n+3})| + \dots + \\ &\quad s^{m-n-2}\,|d\,(x_{m-3},x_{m-2})| + s^{m-n-1}\,|d\,(x_{m-2},x_{m-1})| + s^{m-n-1}\,|d\,(x_{m-1},x_m)| \end{aligned}$$

By using (3), we get

$$\begin{aligned} |d\,(x_n,x_m)| &\leq s\delta^n\,|d(x_0,x_1)| + s^2\delta^{n+1}\,|d(x_0,x_1)| + s^3\delta^{n+2}\,|d(x_0,x_1)| + \dots + s^{m-n-2}\delta^{m-3}\,|d(x_0,x_1)| \\ &\quad + s^{m-n-1}\delta^{m-2}\,|d\,(x_0,x_1)| + s^{m-n}\delta^{m-1}\,|d\,(x_0,x_1)| \\ &= \sum_{i=1}^{m-n} s^i\delta^{i+n-1}\,|d\,(x_0,x_1)|. \end{aligned}$$

Therefore,

$$\begin{aligned} |d\,(x_n,x_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}\,|d\,(x_0,x_1)| \\ &= \sum_{t=n}^{m-n} s^t\delta^t\,|d\,(x_0,x_1)| \\ &\leq \sum_{t=1}^{\infty}(s\delta)^t\,|d\,(x_0,x_1)| \\ &= \frac{(s\delta)^n}{1-s\delta}\,|d\,(x_0,x_1)|. \end{aligned}$$

and hence $|d\,(x_n,x_m)| \leq \frac{(s\delta)^n}{1-s\delta}|d\,(x_0,x_1)| \to 0$ as $m, n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists some u $\in$ X such that $x_n \to u$ as $n \to \infty$.
We show that Su = u. Assuming not, then there exist z $\in$ X such that

$$|d(u, Su)| = |z| > 0. \tag{4}$$

By using the triangular inequality and (1), we get

$$z = d(u, Su) \precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su)$$
$$= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su)$$
$$\precsim sd(u, x_{2n+2}) + s\left\{\lambda d(u, x_{2n+1}) + \frac{\mu\{d(u,Tx_{2n+1})d(x_{2n+1},Tx_{2n+1}) + d(x_{2n+1},Su)d(x_{2n+1},Tx_{2n+1})\}}{1+d(u,Tx_{2n+1})d(x_{2n+1},Su)}\right\}$$
$$= sd(u, x_{2n+2}) + s\lambda d(u, x_{2n+1}) + \frac{s\mu\{d(u,x_{2n+2})d(x_{2n+1},Tx_{2n+2}) + d(x_{2n+1},Su)d(x_{2n+1},Tx_{2n+2})\}}{1+d(u,Tx_{2n+2})d(x_{2n+1},Su)} \tag{5}$$

Taking limit of (5) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (4).
So $|z| = 0$.
Hence Su = u.
Similarly, we can show that Tu = u.

Uniqueness: Let u* be another common fixed point of S and T. Then

$$d(u, u^*) = d(Su, Tu^*)$$
$$\preceq \lambda d(u, u^*) + \frac{\mu\{d(u,Tu*)d(u*,Tu*) + d(u*,Su)d(u*,Tu*)\}}{1+ d(u,Tu*) + d(u*,Su)}$$
$$\preceq \lambda d(u, u^*).$$

This implies that $|d(u, u*)| \preceq |\lambda d(u, u*)|$, a contradiction. So, u = u* which proves the uniqueness of common fixed point in X. this completes the proof.

Corollary 1: Let (X, d) be a complete complex valued b-metric space with the coefficient s $\geq$ 1 and let T : X $\to$ X be a self mappings satisfying:

$$d(Sx, Ty) \precsim \lambda d(x,y) + \frac{\mu\{d(x,Ty)d(y,Ty) + d(y,Tx)d(y,Ty)\}}{1+ d(x,Ty) + d(y,Tx)} \tag{6}$$

for all x, y $\in$ X, where λ, μ are non-negative reals with $s\lambda + \mu < 1$. Then the mapping T has unique common fixed point in X.

Proof: We can prove the theorem by applying Theorem (1) with S = T.

Corollary 2: Let (X, d) be a complete complex valued b-metric space with the coefficient s $\geq$ 1 and let T : X $\to$ X be a self mappings satisfying:

$$d(Tx, T_y^n) \precsim \lambda d(x,y) + \frac{\mu\{d(x,T^n y)d(y,T^n y) + d(y,T^n x)d(y,T^n y)\}}{1+ d(x,T^n y) + d(y,T^n x)} \tag{7}$$

for all x, y $\in$ X, where λ, μ are non-negative reals with $s\lambda + \mu < 1$. Then the mapping T has unique common fixed point in X.

Proof: From theorem (1), we obtain u $\in$ X such that

$$T^n u = u$$

The uniqueness follows from

$$D(Tu, u) = d(TT^n u, T^n u) = d(TT^n u, T^n u)$$

$$\precsim \lambda d(Tu,u) + \frac{\mu\{d(Tu,T^n u)d(u,T^n u) + d(u,T^n Tu)d(u,T^n u)\}}{1 + d(T^u,T^n u) + d(u,T^n T^n u)}$$
$$\precsim \lambda d(Tu,u) + \frac{\mu\{d(Tu,u)d(u,u) + d(u,T^n Tu)d(u,u)\}}{1 + d(T^u,u) + d(u,T^n u)}$$
$$= \lambda d(Tu,u). \tag{8}$$

Taking modulus of (8) and since $\lambda < 1$, we obtain

$$|d(Tu,u)| \le \lambda\ |d(Tu,u)| < |d(Tu,u)|,$$ a contradiction.

So, Tu = u and thus

$$Tu = T^n u = u.$$

Therefore the fixed point of T is unique.

Example: Let X = C. Define a function d : X x X →C such that

$$d(z_1, z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2 \tag{9}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 = ix_2$.

To verify that (X, d) is a complete complex valued b-metric space with s = 2, it is enough to verify the triangular inequality condition.

Let z_1, z_2 and $z_3 \in X$, then

$$d(z_1, z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2$$
$$= |x_1 - x_3 + x_3 - x_2|^2 + i|y_1 - y_3 + y_3 - y_2|^2$$
$$\precsim |x_1 - x_3|^2 + |x_3 - x_2|^2 + 2|x_1 - x_3| + |x_3 - x_2| +$$
$$i[|y_1 - y_3|^2 + |y_3 - y_2|^2\ 2|y_1 - y_3| + |y_3 - y_2|]$$
$$\precsim |x_1 - x_3|^2 + |x_3 - x_2|^2 + |x_1 - x_3|^2 + |x_3 - x_2|^2 +$$
$$i[|y_1 - y_3|^2 + |y_3 - y_2|^2 + i|y_1 - y_3|^2 + |y_3 - y_2|^2]$$
$$= 2\{|x_1 - x_3|^2 + |x_3 - x_2|^2 + i|y_1 - y_3|^2 + |y_3 - y_2|^2]\}$$
$$= 2\,[d(z_1, z_3) + d(z_3, z_2)].$$

Therefore, s = 2.

Now define two self mappings S, T : X → X as follows:

$$Tz = T(x+iy) = \begin{cases} 0 & if\ x,y \in Q \\ 2 & if\ x \in Q^c, y \in Q \\ 2i & if\ x \in Q^c, y \in Q^c \\ 2+2i & if\ x \in Q, y \in Q^c \end{cases}$$

such that S = T and s = x + iy. Let $x = \frac{1}{\pi}$ and y = 0, and since $\lambda \in [1, 0)$, we have d(Tx, Ty) = $d\left(T\frac{1}{\pi}, T0\right) = d(2,0) = 4 \succ \lambda\frac{1}{\pi^2}$

$$= \lambda d\left(T\frac{1}{\pi}, 0\right) + \frac{\mu\{d\left(T\frac{1}{\pi}, T0\right)d(0,T0) + d\left(0, T\frac{1}{\pi}\right)d(0,0)}{1 + d\left(T\frac{1}{\pi}, T0\right) + d\left(0, T\frac{1}{\pi}\right)}$$

Note that $T^n z = 0$ for n > 1, so

$$d(T^n x, T^n y) = 0 \precsim \lambda d(x, y) + \frac{\mu\{d(x,T^n y)d(y,T^n y) + d(y,T^n x)d(y,T^n y)\}}{1 + d(x,T^n y) + d(u,T^n x)} \tag{10}$$

for all x, y $\in$ X and $\lambda, \mu \ge 0$ with $2\lambda + \mu < 1$. So, all conditions of corollary (2) are satisfied to get a unique fixed point 0 of T.

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