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Some Fixed Point Theorems in G – Metric Spaces

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ABSTRACT

In this paper we give some fixed point theorems in G-metric spaces through rational contractive conditions. Our aim is to generalize the theorems in various aspects.

Keywords: Metric space, complete metric space, expansive mapping, G-metric space, G-cauchy space, G-complete metric space and G-continuously metric space.

INTRODUCTION

In 1963, generalization of metric spaces by Ghaler[3,4] introduced the concept of 2-metric space. The result of Ghaler[3,4] was generalized by Dhage[2], in 1984, which are called D-metric spaces. The situation for the D-metric space is quite different from the 2-metric spaces. Geometrically, a D-metric D (x, y, z) represents the perimeter of the triangle with vertices x, y and z in Rz.

A number of fixed point theorems have been proved for 2-metric and D-metric spaces. However, Hsiao[5] shows that all such theorems are trivial in the sense that the iterations of f are all collinear. It was shown that certain theorems involving Dhage's D-metric spaces are flawed and most of the results claimed by Dhage and others are invalid.

In 2005, Mustafa and Sims[6-8] introduced a more appropriate and robust notion of a generalized metric space known as G-metric spaces, there are several interesting results are published by various mathematicians[1,9-12].

Definition 1: Let X be any non-empty set and let $d: x \otimes x \to [0, \infty)$ be a function satisfying following conditions:
(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$
(iii) $d(x, y) = d(y, x)$
(ii) $d(x, y) = d(x, z) + d(z, y) \forall x, y, z \in X$.
If d is distance function on X. Then the pair (X, d) is called metric space.

Definition 2: A sequence $\{x_n\}$ in metric space (x, d) is called Cauchy sequence if for a given $\in >0$ there exists a number $n_0 \in N$ such that $\forall$ m, n> n_0, $d(x_m, x_n) < \in$.

Definition 3: A sequence $\{x_n\}$ in metric space (x, d) is convergent to $x \in X$ if $\lim_{n \to \infty} x_n = x$. In this case x is called a limit of the sequence $\{x_n\}$.

Definition 4: A metric space (x, d) is called complete if every Cauchy sequence is convergent.

Definition 5: Let (X, d) be a metric space and $T : X \to X$ be a mapping then T is said to be expansive mapping if for every x, y $\in$ X there exists a number r > 1 such that
$D(T_x, T_y) \geq r d(x,y)$

Definition 6: Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following axioms:
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $G(x, x, y) > 0$ for all x, y $\in$ X with $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z $\in$ X with $y \neq z$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$
(Symmetry in all three variables)
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, a $\in$ X
(Rectangle inequality)
Then the function G is called a generalized metric or G-metric on X and (X, G) is called a G-metric space.

Definition 7: Let (X, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X, a point x $\in$ X is said to the limit of the sequence $\{x_n\}$, if $lim_{n\to\infty} G(x, x_n, x_m) = 0$. Then $\{x_n\}$ is convergent to X.

Proposition 1: Let (X, G) be a G-metric space. Then for any x, y, z, a $\in$ X, if follows that
(i) If $G(x, y, z) = 0$ then $x = y = z$
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$
(iii) $G(x, y, y) \leq 2G(y, x, x)$
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$
(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$
(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Proposition 2: Let (X, G) be a G-metric space, then for a sequence $\{x_n\} \subseteq X$ and a point x $\in$ X.

The following are equivalent
(i) $\{x_n\}$ is G-convergent to x.
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 8: Let (X, G) be a G-metric space, then the sequence $\{x_n]$ is said to be G-Cauchy if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$ i.e. $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 9: A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 3: Let (X, G), (X', G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if only if it is G-sequentially continuous at x; i.e. whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)]$ is G- convergent to f(x).

Theorem 1: Let (X, G) be a complete G-metric space and let $T: X \to X$ be a mapping which satisfies the following condition:

$$G(Tx, Ty, Tz) \leq \alpha \frac{G(x,Ty,Ty)+ G(x,Tz,Tz)}{2} + \beta \frac{G((x,Ty,Ty)[G(x,Ty,Ty)+G(x,Tz,Tz)+G(y,Tx,Tx)+G(z,Tx,Tx)}{2[G((x,Ty,Ty)+G(y,Tx,Tx)]} \quad (1)$$

For all $x, y, z \in X$, where $0 \leq (\alpha+\beta) < \frac{1}{2}$,
Then T has a unique fixed point u and T is G-continuous at u.

Proof : let $x_0 \in x$ be an arbitrary point and define the sequence $\{x_n\}$ by $Tx_0 = x_1$, $Tx_1 = x_2$, $Tx_2 = x_3$, … … … … $Tx_n = x_{n+1}$.

Then by (1), we get

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \alpha \frac{G(x_{n-1},Tx_n,Tx_n)+ G(x_{n-1},Tx_n,Tx_n)}{2} \\ &+ \beta \frac{G(x_{n-1},Tx_n,Tx_n)\begin{bmatrix} G(x_{n-1},Tx_n,Tx_n)+G(x_{n-1},Tx_n,Tx_n) \\ +G(x_n,Tx_{n-1},Tx_{n-1})+G(x_n,Tx_{n-1},Tx_{n-1}) \end{bmatrix}}{2[G(x_{n-1},Tx_n,Tx_n)+G(x_{n-1},Tx_n,Tx_n)]} \\ &\leq \alpha \frac{G(x_{n-1},x_{n+1},x_{n+1})+ G(x_{n-1},x_{n+1},x_{n+1})}{2} \\ &+ \beta \frac{G(x_{n-1},x_{n+1},x_{n+1})\begin{bmatrix} G(x_{n-1},x_{n+1},x_{n+1})+G(x_{n-1},x_{n+1},x_{n+1}) \\ +G(x_n,x_n,x_n)+G(x_n,x_n,x_n) \end{bmatrix}}{2[G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)]} \end{aligned}$$

$$\leq \alpha\ G(x_{n-1}, x_{n+1}, x_{n+1}) + \beta G(x_{n-1}, x_{n+1}, x_{n+1}) \quad (2)$$

But by (G_5), we get

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})$$

So, (2) becomes

$G(x_n, x_{n+1}, x_{n+1}) \le (\alpha + \beta)[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})]$

$G(x_n, x_{n+1}, x_{n+1}) \le \frac{(\alpha+\beta)}{(1-\alpha-\beta)} G(x_{n-1}, x_n, x_n)$

Let $K = \frac{(\alpha+\beta)}{(1-\alpha-\beta)} < 1$

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le K G(x_{n-2}, x_{n-1}, x_{n-1}) \quad (3)$$

On further decomposing we can write

$$G(x_{n-1}, x_n, x_n) \le K G(x_{n-2}, x_{n-1}, x_{n-1}) \quad (4)$$

By combination of (3) and (4) we have

$G(x_n, x_{n+1}, x_{n+1}) \le K^2 G(x_{n-2}, x_{n-1}, x_{n-1})$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \le K^n G(x_0, x_1, x_1) \quad (5)$$

Then for all n, m $\in$ N, n < m we have

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) \\ &+ G(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \dots + G(x_{m-1}, x_m, x_m) \\ &\le (K^n + K^{n+1} + \dots \dots + K^{m-1})\, G(x_0, x_1, x_1) \\ &\le \frac{K^n}{1-k} G(x_o, x_1, x_1) \end{aligned}$$

Taking limit as n, m $\to \infty$, we get

lim $G(x_n, x_m, x_m) = 0$.

Therefore $\{x_n\}$ is G-Cauchy sequence, hence G-convergent, since X is G-complete metric space so $\{x_n\}$ is G-convergence to u.

Suppose that Tu $\neq$ u, then

From (1) we have

$$\begin{aligned} G(x_n, T_u, T_u) &= G(Tx_{n-1}, Tu, T_u) \\ &\le \alpha \frac{G(x_{n-1},Tu,Tu) + G(x_{n-1},Tu,Tu)}{2} \\ &+ \beta \frac{G(x_{n-1},Tu,Tu)\left[\begin{smallmatrix} G(x_{n-1},Tu,Tu)+G(x_{n-1},Tu,Tu) \\ +G(u,Tx_{n-1},Tx_{n-1})+G(u,Tx_n,Tx_{n-1},Tx_{n-1}) \end{smallmatrix}\right]}{2[G(x_{n-1},Tu,Tu)+G(u,Tx_{n-1},Tx_{n-1})]} \\ &\le \alpha \frac{G(x_{n-1},Tu,Tu) + G(x_{n-1},Tu,Tu)}{2} \end{aligned}$$

$$+ \beta \frac{G(x_{n-1},Tu,Tu)[G(x_{n-1},Tu,Tu)+G(x_{n-1},Tu,Tu)+G(x_{n-1},Tu,Tu)+G(u,x_n,x_n)++G(u,x_n,x_n)]}{2[G(x_{n-1},Tu,Tu)+G(u,Tx_{n-1},Tx_{n-1})]}$$

Taking the limit of both sides as n $\to \infty$ and using the fact that G is continuous. Then

$G(u, Tu, Tu) \le (\alpha + \beta)\, G(u, Tu, Tu)$

This is a contradiction. Hence u = Tu

Uniqueness: Suppose that u and v are two fixed point for T.

Then (1) implies that

$$\begin{aligned} G(u, v, v) &\le G(Tu, Tv, Tv) \\ &\le \alpha \frac{G(u,Tv,Tv) + G(u,Tv,Tv)}{2} \end{aligned}$$

$+ \ \beta \frac{G(u,Tv,Tv)[G(u,Tv,Tv)+G(u,Tv,Tv)+G(v,Tu,Tu)+G(v,Tu,Tu)]}{2[G(u,Tv,Tv)+G(v,Tu,Tu)]} \leq \alpha G(u, v, v) + \beta G(u, v, v)$

$G(u, v, v) \leq (\alpha + \beta)\, G(u, v, v)$

$\Rightarrow G(u, v, v) = 0$

$\Rightarrow u = v$

To show that T is G-continuous at u. Let $\{y_n\}$ be a sequences converges to u in (X, G). Then we can deduce that

$G(u, Ty_n, Ty_n) = G(Tu \ , Ty_n, Ty_n)$

$\leq \alpha \frac{G(u,Ty_n,Ty_n) + G(u,Ty_n,Ty_n)}{2}$

$+ \beta \frac{G(u,Ty_n,Ty_n)[G(u,Ty_n,Ty_n + G(u,Ty_n,Ty_n) + G(y_n,Tu,Tu) + G(y_n,Tu,Tu)]}{2G(u,Ty_n,Ty_n) + G(y_n,Tu,Tu)}$

$G(u, Ty_n, Ty_n) \leq (\alpha + \beta)\, G(Tu \ , Ty_n, Ty_n)$

$[1 - (\alpha + \beta)]\, G(u, Ty_n, Ty_n) \leq 0$

$G(u, Ty_n, Ty_n) \leq 0$

Taking the limit as $n \to \infty$ from which we see that $G(u, Ty_n, Ty_n) \to 0$ and so, by proposition (2) we have that the sequence Ty_n is G-convergent to Tu = u therefore proposition (3) implies that T is G-continuous at u.

Theorem 2: Let (X, G) be complete G-metric space and let $T : X \to X$ be a mapping satisfying the condition:

$$G(Tx, Ty, Tz) \leq \alpha \min \begin{Bmatrix} G(x, Tx, Tx), G(y, Ty, Ty), \\ G(z, Tz, Tz), G(x, y, z) \end{Bmatrix} + \beta \left[\frac{G(x,Tx,Tx) + G(y,Tx,Tx), + G(y,Ty,Ty)}{1 + G(x,Tx,Tx)G(y,Tx,Tx)G(x,Ty,Ty)}\right] \qquad (1)$$

For all x, y, z $\in$ X where $\alpha, \beta \geq 0$ and $\alpha + 3\beta < 1$.

Then T has unique fixed point u and T is G-continuous at u.

Proof: Let x0 $\in$ X be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = Ty_{n-1}$. Then by (1), we have

$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$

$\leq \alpha \min \begin{Bmatrix} G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \ , G(x_n, Tx_n, Tx_n), \\ G(x_n, Tx_n, Tx_n), G(x_{n-1}, x_n, x_n) \end{Bmatrix}$

$+\beta \left[\frac{G(x_{n-1},Tx_{n-1},Tx_{n-1}) + G(x_n,Tx_{n-1},Tx_{n-1}) + G(x_{n-1},Tx_n,Tx_n)}{1 + GG(x_{n-1},Tx_{n-1},Tx_{n-1})G(x_n,Tx_{n-1},Tx_{n-1})G(x_{n-1},Tx_n,Tx_n)}\right]$

$\leq \alpha \min \begin{Bmatrix} G(x_{n-1}, x_n, x_n) \ , G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n) \end{Bmatrix}$

$+\beta \left[\frac{G(x_{n-1},x_n,x_n) + G(x_n,x_n,x_n) + G(x_n,x_{n+1},x_{n+1})}{1 + G(x_{n-1},x_n,x_n)G(x_n,x_n,x_n)G(x_n,x_{n+1},x_{n+1})}\right]$

$\leq \alpha \min \{G(x_{n-1}, x_n, x_n) \ , G(x_n, x_{n+1}, x_{n+1})\}$

$+\beta \, [G(x_{n-1}, x_n, x_n) + \ G(x_{n-1}, x_{n+1}, x_{n+1})] \qquad (2)$

Here two cases arise:

Case I: If $\min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n)$

Then condition (2) reduces to

$$G(x_n, x_{n+1}, x_{n+1}) \le \alpha G(x_{n-1}, x_n, x_n) + \beta\, [G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})] \quad (3)$$

But by (G_5), we get

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})$$

So, (3) becomes

$$G(x_n, x_{n+1}, x_{n+1}) \le \alpha G(x_{n-1}, x_n, x_n) + \beta\, [2G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})]$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{(\alpha+2\beta)}{(1-\beta)} G(x_{n-1}, x_n, x_n)$$

Let $K = \frac{(\alpha+2\beta)}{(1-\beta)} < 1$

$$G(x_n, x_{n+1}, x_{n+1}) \le K G(x_{n-1}, x_n, x_n)$$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \le K^n G(x_0, x_1, x_1)$$

Then for all n, m $\in$ N, n < m we have

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\le (K^n + K^{n+1} + \ldots\ldots\ldots\ldots + K^{m-1})\, G\,(x_0, x_1, x_1)$$

$$\le \frac{K^n}{1-K} G\,(x_0, x_1, x_1)$$

Taking limit as n, m $\to \infty$, we get

$$\lim G\,(x_0, x_1, x_1) = 0$$

Therefore, $\{x_n\}$ is G-Cauchy sequence. Hence G-convergent, since X is G-complete metric space so X is G-converges to u.

Suppose that Tu $\neq$ u, then

From (1) we have

$$G(u, Tu, Tu) = G(x_n, Tu, Tu) = G(Tx_{n+1}, Tu, Tu)$$

$$\le \alpha \min \begin{Bmatrix} G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(u, Tu, Tu), \\ G(u, Tu, Tu), G(x_{n-1}, u, u) \end{Bmatrix}$$

$$+\beta \left[\frac{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(u, Tx_{n-1}, Tx_{n-1}) + G(x_{n-1}, Tu, Tu)}{1 + G(x_{n-1},, x_n, x_n)\, G(u, Tx_{n-1}, Tx_{n-1}) G(x_{n-1}, Tu, Tu)}\right]$$

$$\le \alpha \min \begin{Bmatrix} G(x_{n-1}, x_n, x_n), G(u, Tu, Tu), \\ G(u, Tu, Tu), G(x_{n-1}, u, u) \end{Bmatrix}$$

$$+\beta \left[\frac{G(x_{n-1}, x_n, x_n) + G(u, x_n, x_n) + G(x_{n-1}, Tu, Tu)}{1 + G(x_{n-1},, x_n, x_n)\, G(u, x_n, x_n) G(x_{n-1}, Tu, Tu)}\right]$$

Taking the limit as n $\to \infty$ and using the fact that G is continuous. Then

G (u, Tu, Tu) $\le \beta$ G (u, Tu, Tu)

This is a contraction. Since $\beta < 1$.

$\Rightarrow$G (u, Tu, Tu) = 0

So, u = Tu

Uniqueness: Suppose that u and v are two fixed point for T.
Then (1) implies that

$$G(u, v, v) = G(Tu, Tv, Tv)$$
$$\leq \alpha \min \begin{Bmatrix} G(u, Tu, Tu), G(v, Tv, Tv), \\ G(v, Tv, Tv), G(u, v, v) \end{Bmatrix}$$
$$+\beta \left[\frac{G(u,Tu,Tu)+G(v,Tu,Tu)+G(u,Tv,Tv)}{1+G(u,Tu,Tu)\ G(v,Tu,Tu)G(u,Tv,Tv)}\right]$$
$$\leq \beta\, G(v, u, u)$$
$$G(u, v, v) \leq 2\beta G(u, v, v)$$

Which is a contradiction. Therefore, G (u, v, v) = 0
Hence u = v.
To show that T is G-continuous at u. Let $\{y_n\}$ be a sequence converges to u in (X, G).
Then we can conclude that

$$G(u, Ty_n, Ty_n) = G(Tu, Ty_n, Tx_n)$$
$$\leq \alpha \min \begin{Bmatrix} G(u, Tu, Tu), G(y_n, Ty_n, Ty_n), \\ G(y_n, Ty_n, Ty_n), G(u, y_n, y_n) \end{Bmatrix}$$
$$+\beta \left[\frac{G(u,Tu,Tu)+G(y_n,Tu,Tu)+G(u,Ty_n,Ty_n)}{1+G(u,Tu,Tu)G(y_n,Tu,Tu)G(u,Ty_n,Ty_n)}\right]$$
$$G(u, Ty_n, Ty_n) \leq \frac{\beta}{1-\beta}\ G(y_n, u, u)$$

Taking the limit as $n \to \infty$ from which we see that $G(u, Ty_n, Ty_n) \to 0$ and so, by proposition we have the sequence Ty_n is G-convergent to Tu = u therefore proposition implies that T is G-continuous at u.

There exists a number r > 1 such that d (Tx, Ty) ≥ rd(x, y).

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