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Fixed Point Theorems in Fuzzy Metric Space by using Commutativity of Type (A_g)

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ABSTRACT

The purpose of this paper is to extend the study of non-compatible maps in fuzzy metric space by using the notion of R-weakly commutativity of type (A_g) in fuzzy metric space.

Keywords: Fixed point, Fuzzy metric space, E.A. property, non-compatible mappings, R-weakly commutativity.

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh[14]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek[4], George and Veeramani[1] modified the notion of Fuzzy metric space with the help of continuous t-norms. Choudhary[3] introduced mutually contractive sequence of self maps and proved a fixed point theorem. Kramosil and Michalek[11] introduced the notion of Cauchy sequences in a fuzzy metric space and proved the well known fixed point theorem of Banach[1]. Recently Aamri and Moutawakil[15] introduced the property (E.A) and thus generalized the concept of non-compatible maps. Many authors have studied the fixed point theory in the fuzzy metric space and numbers of fixed point theorems have been obtaining in fuzzy metric space by using the contractive condition of self-mappings. The results obtained in the metric fixed point theory by using the notion of non-compatible maps or the property (E.A) is very interesting.

The aim of the present paper is to extend the study of non-compatible maps in Fuzzy metric space by using the notion of R-weakly commutativity of type (A_g) in fuzzy metric space. Our Result will generalize the result of many others.

2. PRELIMINARIES

Definition 2.1 [16] A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever

$a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.
Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$

Definition 2.2[1] The 3-tuple (X, M, $*$) is called a fuzzy metric space (FM-space) if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$.
2.2.1 $M(x, y, 0) > 0$
2.2.2 $M(x, y, t) = 1, \forall\, t > 0$ iff $x = y$
2.2.3 $M(x, y, t) = M(y, x, t)$,
2.2.4 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
2.2.5 $M(x, y, \ .): [0, \infty] \to [0,1]$ is continuous.

Example 2.1 Induced Fuzzy metric space[6]
Let (X, d) be a metric space denote $a * b = ab$ for all $a, b \in [0,1]$ and let M be a fuzzy set on $X^2 \times (0,1)$ defined as follow:
$$M(x, y, t) = \frac{t}{t + d(x,y)}$$
Then $(X, M,*)$ is a Fuzzy metric space.

Remark 2.1 since $*$ is continuous, it follows from (2.2.4) that the limit of a sequence in FM-space is uniquely determined.

Remark 2.2 Note that the above example holds even with the t-norm $a * b = \min\{a, b\}$

Definition 2.3 [6] Let $(X, M,*)$ be a fuzzy metric space then

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0 \in N$ such that
$$M(x_n, x, t) > 1 - \varepsilon$$
(b) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete

Definition 2.4 [8] Let A and B map from A fuzzy metric space $(X, M,*)$ into itself. The maps A and B are said to be compatible (or asymptotically commuting) if for all $t > 0$
$$lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$$
Where $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = z$ for some $z \in X$

Definition 2.5 [10] Two mappings A and B of a fuzzy metric space $(X, M,*)$ into itself are R-weakly commuting provided there exists some real number R such that
$$M(ABx, BAx, t) \geq M(Ax, Bx, t/R)$$

Definition 2.6 Two mappings A and S of a fuzzy metric space into itself are R-weakly commuting of type (A_g) provided there exists some real number R such that
$M(AAx, SAx, t) \geq M(Ax, Sx, t/R)$ for each $x \in X$ and $t > 0$

Definition 2.7 [15] Let f and g be two self mappings of a fuzzy metric space$(X, M,*)$. We say that A and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$lim_{n\to\infty} f x_n = lim_{n\to\infty} g x_n = t \text{ For some } t \in X$$

Lemma 2.1[20] In A fuzzy metric space X, $M(x, y, .)$ is non decreasing for all $x, y \in X$

Lemma 2.2[20] Let $(X, M,*)$ be a fuzzy metric space if there exist $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$, then $x = y$.

Proof: Since, $M(x, y, kt) \geq M(x, y, t)$
Then we have,

$$M(x, y, t) \geq M(x, y, \frac{t}{k})$$

By repeated application of above inequality as we have

$$M(x, y, t) \geq M\left(x, y, \frac{t}{k}\right) \geq M\left(x, y, \frac{t}{k^2}\right) \geq - - - - M\left(x, y, \frac{t}{k^n}\right) \geq - - - - - - - - - -$$

for $n \in N$ which tends to 1 as $n \to \infty$ respectively
Thus $M(x, y, t) = 1$ for all $t > 0$ and we get $x = y$.

3. MAIN RESULTS

Theorem 3.1 Let f and g be point wise R-weakly commuting self maps of type (A_g) of a fuzzy metric space $(X, M,*)$ such that

(3.1.1) $fX \subset gX$,

(3.1.2) $M(fx, fy, th) \geq \{\frac{(M(fx,gx,t)M(fy,gy,t)+M(fx,gy,t)M(fy,gx,t))}{1+M(gx,gy,t)}\}$; $0 < h < 1, t > 0$.

If f and g satisfy the property (E.A) and the range of either of f or g is complete subspace of X, then f and g have a unique common fixed point.

Proof: Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$lim_{n\to\infty} f x_n = lim_{n\to\infty} g x_n = p \quad \text{for some p in X.}$$

Since $p \in fX$ and $fX \subset gX$, there exists some point u in X such that $p = gu$ where $p = lim_{n\to\infty} g x_n$

If $fu \neq gu$, the inequality (3.2) implies

$$M(fx_n, fu, th) \geq \{\frac{(M(fx_n, gx_n, t)M(fu, gu, t) + M(fx_n, gu, t)M(fu, gx_n, t))}{1 + M(gx_n, gu, t)}\}$$

On letting $n \to \infty$ yields

$$M(gu, fu, th) \geq \{\frac{(M(gu, gu, t)M(fu, gu, t) + M(gu, gu, t)M(fu, gu, t))}{1 + M(gu, gu, t)}\}$$

$$\Rightarrow M(gu, fu, th) \geq \{\frac{(M(fu,gu,t)+M(fu,gu,t))}{1+1}\}$$

$$\Rightarrow M(gu, fu, th) \geq M(gu, fu, t)$$

Therefore by Lemma 2.2, $gu = fu$.
Since f and g are R-weak commutating of type (A_g), there exists R > 0 such that
$M(ffu, gfu, t) \geq M(fu, gu, t/R)$,

$$\Rightarrow M(ffu, gfu, t) = 1$$

That is, $ffu = gfu$ and $ffu = fgu = gfu = ggu$.
If $fu \neq ffu$, putting $x = u, y = fu$ in inequality (3.1.2), we get

$$M(f\mathrm{u}, ffu, t\mathrm{h}) \geq \{\frac{(M(f\mathrm{u}, g\mathrm{u}, t)M(ffu, gfu, t) + M(gu, ffu, t)M(ffu, g f\mathrm{u}, t))}{1 + M(gu, gfu, t)}\}$$

$$M(f\mathrm{u}, ffu, t\mathrm{h}) \geq \left\{\frac{(1 + M(fu, ffu, t))}{1 + M(g\mathrm{u}, gfu, t)}\right\} = 1 > M(fu, ffu, t)$$

which is a contradiction.
Hence, $fu = ffu \; and \; fu = ffu = fgu = gfu = ggu.$
Hence fu is a common fixed point of f and g. The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$ and the theorem is proved.

Theorem 3.2: Let f and g be non-compatible point-wise R-weakly commuting self-maps of type$(\mathrm{A_g})$ of a fuzzy metric space $(\mathrm{X}, \mathrm{M}, *)$ such that
(3.2.1) $fX \subset gX$,
(3.2.2) $M(fx, fy, t\mathrm{h}) \geq \{\frac{(M(fx,gx,t)M(fy,gy,t)+M(fx,gy,t)M(fy,gx,t))}{1+M(gx,gy,t)}\}$ where $0 < \mathrm{h} < 1, \mathrm{t} > 0$
If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point and the fixed point is the point of discontinuity.

Proof: Since f and g are non -compatible maps, there exists a sequence $\{\mathrm{x_n}\}$ in X such that

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = p \text{ for some p in X.} \qquad (1)$$

But either $lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or lim does not exist.
Since $p \in fX$ and $fX \subset gX$, there exists some point u in X such that $p = gu$ where $p = lim_n gx_n$.
If $fu \neq gu$, then putting $x = x_n, \mathrm{y} = u$ in inequality (2.2)

$$M(fx_n, fu, t\mathrm{h}) \geq \{\frac{(M(fx_n, gx_n, t)M(fu, gu, t) + M(fx_n, gu, t)M(fu, gx_n, t))}{1 + M(gx_n, gu, t)}\}$$

On letting $\mathrm{n} \to \infty$ yields

$$M(gu, fu, t\mathrm{h}) \geq \{\frac{(M(gu, gu, t)M(fu, gu, t) + M(gu, gu, t)M(fu, gu, t))}{1 + M(gu, gu, t)}\}$$

$$\Rightarrow M(gu, fu, t\mathrm{h}) \geq \{\frac{(M(fu,gu,t)+M(fu,gu,t))}{1+1}\}$$

$$\Rightarrow M(gu, fu, t\mathrm{h}) \geq M(gu, fu, t)$$

Therefore by Lemma 2.2, $gu = fu$.

Since f and g are R-weakly commutating of type(A_g), there exists R > 0 such that

$$M(ffu, gfu, t) \geq M(fu, gu, t/R)$$
$$\Rightarrow M(ffu, gfu, t) = 1$$

That is $fu = gfu \; and \; ffu = fgu = gfu = ggu$.

If $fu \neq ffu$, putting $x = u, y = fu$ in inequality (3.2.2), we have

$$M(f\text{u}, ffu, t\text{h}) \geq \{\frac{\big(M(f\text{u}, g\text{u}, t)M(ffu, gfu, t) + M(gu, ffu, t)M(ffu, g\text{fu}, t)\big)}{1 + M(g\text{u}, gfu, t)}$$

$$\Rightarrow M(f\text{u}, ffu, t\text{h}) \geq \left\{\frac{\big(1 + M(fu, ffu, t)\big)}{1 + M(g\text{u}, gfu, t)}\right\} = 1 > M(f\text{u}, ffu, t)$$

which is a contradiction.

Hence $fu = ffu \; and \; fu = ffu = fgu = gfu = ggu$

Hence fu is a common fixed point of f and g. The case when fX is complete subspace of X is similar to the above case since $fX \subset gX$.

Now we have to show that f and g are discontinuous at the common fixed point $p = fu = gu$.

If possible, suppose f is continuous, then considering the sequence $\{x_n\}$ of (1) we get

$$lim_{n\to\infty} ffx_n = fp = p.$$

And R-weak commutatively of type (A_g)implies that

$$M(ffx_n, gfx_n, t) \geq M(fx_n, gx_n, t/R) = 1$$

which on letting $n \to \infty$ yields $lim_n gfx_n = fp = p$.

This, in turn, yields $lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$.

This contradicts the fact that $lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either non zero or does not exist for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at the fixed point.

Now suppose that g is continuous, then for sequence $\{x_n\}$ of (1), we get

$lim_{n\to\infty} gfx_n = gp = p$ and $lim_{n\to\infty} ggx_n = gp = p$.

Putting $x = x_n, \; y = gx_n$ in inequality (3.2.2)

$$M(fx_n, fgx_n, t\text{h}) \geq \{\frac{\big(M(fx_n, gx_n, t)M(fgx_n, ggx_n, t) + M(fx_n, ggx_n, t)M(fgx_n, gx_n, t)\big)}{1 + M(gx_n, ggx_n, t)}\}$$

Which on letting $n \to \infty$ yields

$$M(p, fgx_n, t\text{h}) \geq \frac{\big(M(p, p, t)M(fgx_n, p, t) + M(p, p, t)M(fgx_n, p, t)\big)}{1 + M(p, p, t)}\}$$

$$\Rightarrow M(f\text{u}, ffu, t\text{h}) \geq \left\{\frac{\big(1 + M(fgx_n, p, t)\big)}{1 + M(p, p, t)}\right\} = 1 > M(fgx_n, p, t)$$

which is a contradiction unless $lim_n fgx_n = gp$.

But $lim_{n\to\infty} gfx_n = gp$ contradicts the fact that $lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either non-zero or does not exist for the sequence $\{x_n\}$ of (1).

Thus both f and g are discontinuous at their common fixed point and the theorem is proved.

Now we give an example to illustrate the above theorem.

Example 3.1 Let $X = [2,20]$ and M be the usual metric on $(X, *)$.
Define $f, g: X \to X$ as $f(x) = 2\ if\ x = 2\ or > 5\ and\ f(x) = 6\ if\ 2 < x \leq 5$;
$g(2) = 2, g(x) = x + 4$ if $2 < x \leq 5$ and $g(x) = \frac{(4x+10)}{15}$ if $x > 5$
Also we define $M(fx, gy, t) = \frac{t}{[t+d(fx,gy)]}$ for all x, y in X and $t > 0$ then f and g satisfy all the conditions of the above theorem and have a common fixed point at $x = 2$

In this example $fX = \{2\} \cup \{6\} and\ gX = \{2,6\} \cup \{7 \ldots\ldots\}$ it may be seen that $fX \subset gX$. It can also be verified that f and g are pointwise R-weakly commuting maps of type (A_g) and satisfy the (E.A) property.
To see that f and g are noncompatible, let us consider a sequence$\{x_n = 5 + \frac{1}{n} : n > 1\}$ then $lim_{n\to\infty} fx_n = 2$, $lim_{n\to\infty} gx_n = 2$, $lim_{n\to\infty} fgx_n = 6$ and $lim_{n\to\infty} gfx_n = 2$.
Hence f and g are non-compatible.

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