Journal of Computer and Mathematical Sciences, *Vol.8(8), 373-379 August 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Fixed Point Theorem on Fuzzy Metric Space Using Reciprocally Continuous

B. Vijayabasker Reddy[1] and V. Srinivas[2]

[1]Department of Mathematics,
Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad-501 301 INDIA.
[2]Department of Mathematics,
University college of Science Saifabad, Osmania University Hyderabad, INDIA.
email: basker.bonuga@gmail.com, srinivaveladi@yahoo.co.in

(Received on: August 27, Accepted: August 30, 2017)

ABSTRACT

The aim this paper is to prove a unique common fixed point theorem in fuzzy metric space using the concept of reciprocally continuous and compatible mappings with weakly compatible mappings and This theorem generalizes the Theorem proved by Bijendra Singh *et al.*,[2].

AMS (2000) Mathematics Classification: 54H25, 47H10

Keywords: Fixed point, fuzzy set, fuzzy metric space, self maps, weakly compatible mappings, reciprocally continuous.

1. INTRODUCTION

The notion of the fuzzy sets was introduced by Zadeh[1], in recent times many authors developed the theory of fuzzy sets and applications. Grabic[4] developed the Banach contraction principle in fuzzy version. George and Veeramani[3] have shown that every metric induces a fuzzy metric. B singh and jain studied the notion of weakly compatible maps in fuzzy metric space and also proved theorem on it. Pant introduced the idea of reciprocally continuous mapings in metric spaces[13].

DEFINITIONS AND PRELIMINARIES

Definition 1.1: A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative

(ii) * is continuous
(iii) a*1=a for all $a\in[0,1]$
(iv) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all a, b,c,d $\in[0,1]$

Definition 1.2: A 3-tuple (X, M,*) is said to be fuzzy metric space if X is an arbitrary set,* is continuous t- norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and s,t>0

(FM-1) M(x,y,0)=0
(FM-2) M(x,y,t)=1 for all t>0 if and only if x=y
(FM-3) M(x,y,t)= M(y,x,t)
(FM-4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$
(FM-5) M(x,y,.) : $[0,\infty) \to [0,1]$ is left continuous

The function M(x, y,t) denote the degree of nearness between x and y with respect to 't'. We observe M(x,y,t)=1 when x=y for all t>0 and M(x,y,t) =0 when $t\to\infty$.

Some of the properties of fuzzy metric space and examples are given in paper George and Veeramani.

Example 1.3 (Induced fuzzy metric space**):** Let (X,d) be a metric space defined a*b=min{a,b} for all $x,y\in X$ and t>0,

$$M(x,y,t) = \frac{t}{t+d(x,y)} \qquad ----(a)$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric *M* induced by metric *d* the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on X satisfying (a).

Definition 1.4: Let (X, M,*) be a fuzzy metric space then a sequence $<x_n>$ in X is said to be convergent to a point $x\in X$, if for each $\in>0$,t>0 there exists $n_0\in N$ such that $M(x_n,x,t)>1-\in$ $\forall n\ge n_0$

Definition 1.5: A sequence $<x_n>$ in fuzzy metric space (X,M,*) is said to be Cauchy sequence iff for each $\in>0$,t>0 there exists $n_0\in N$ such that $M(x_n,x_m,t)>1-\in$ $\forall n, m\ge n_0$

Definition 1.6: A fuzzy metric space (X, M,*) is said to be complete if every Cauchy sequence is convergent to a point in X.

Lemma 1.7: For all $x,y\in X$, M(x,y,.) is non decreasing.

Lemma 1.8: let (X,M,*)be a fuzzy metric space if there exists $k\in(0,1)$ such that $M(x,y,kt)\ge M(x,y,t)$ then x=y.

Proposition 1.9: In the fuzzy metric space (X, M,*) if $a*a\ge a$ for all $a\in[0,1]$ then a*b= min{a,b}

Definition 1.10: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be compatible mappings if $\lim_{n\to\infty} M(STx_n, TSx_n, t)=1$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 1.11: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be weakly compatible if they commute at their coincidence point. i.e. if Su=Tu for some u∈X then STu=TSu.

Two compatible maps are weakly compatible maps but converse need not true..

Example: Let X= [0,1], $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ where $d(x,y)=|x-y|$

$$Ax = \begin{cases} \frac{1}{8} & if\ 0 \le x \le \frac{1}{8} \\ \frac{1}{6} & if\ \frac{1}{8} < x \le 1 \end{cases} \qquad Sx = \begin{cases} \frac{1}{4} - x & if\ 0 \le x \le \frac{1}{8} \\ \frac{1}{4} & if\ \frac{1}{8} < x \le 1 \end{cases}$$

for this, take a sequence $X_N = \left(\frac{1}{8} - \frac{1}{n}\right)$ for N≥1, then $\lim_{n\to\infty} AX_N = \lim_{n\to\infty} SX_N = 1/8$ and

$\lim_{n\to\infty} ASx_n = 1/6$ also $\lim_{n\to\infty} SAx_n = 1/8$ So that $\lim_{n\to\infty} M(ASx_n, SAx_n, t) \neq 1$.

From the example given above,the pair (A, S) is weakly compatible but not compatible

Definition 1.12: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be reciprocally continuous on X if $\lim_{n\to\infty} STxn = Sz$ and $\lim_{n\to\infty} TSxn = Tz$ whenever $\lim_{n\to\infty} Sxn = \lim_{n\to\infty} Txn = z$.

The following theorem proved by Bijendra Singh *et al.*,[2] in 2005.

2.0 Theorem: Let A, B, S and T are self maps of a complete Fuzzy metric space (X,M,*) with continuous t-norm * defined by a*b=min {a, b} ,a, b∈[0, 1], satisfying the conditions

2.0.1 $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

2.0.2 S and T are continuous

2.0.3 The pairs (A, S) and (B, T) compatible pairs of maps

2.0.4 $[M(Ax,By,kt)] \ge \min\{M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,2t), M(Ax,Ty,t)\}$

for all x, y in X, k∈(0,1),t>0, lim M(x, y ,t)→1 as t→∞

then A, B, S and T have a unique common fixed point z in X
Now the generalization of the Theorem 2.0 as follows

2.1 Theorem : Let A, B, S and T are self maps of a complete Fuzzy metric space (X, M,*) with continuous t-norm * defined by a*b=min {a,b} and a, b$\in$[0,1], satisfying the conditions

2.1.1 A(X)$\subseteq$T(X) and B(X) $\subseteq$ S(X)
2.1.2 The pair (A, S) is reciprocally continuous and compatible
And
The pair (B, T) is weakly compatible

2.1.3 $[M(Ax,By,kt)] \geq \min\{M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,2t), M(Ax,Ty,t)\}$

for all x,y in X, k$\in$(0,1),t>0 lim M(x,y,t)$\to$1 as t$\to\infty$
then A, B, S and T have a unique common fixed point z in X
Following lemma playing important role in Main Theorem 2.1

2.2.4 Lemma: Let A,B,S and T be self mappings from a complete fuzzy metric space (X,M,*) into itself satisfying the conditions (2.1.1) and (2.1.3)Then the sequence $\{y_n\}$ defined by $y_{2n}=Ax_{2n}= Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+1} = Sx_{2n+2}$ for n ≥ 0 relative to four self maps is a Cauchy sequence in X.

Proof of Lemma 2.2.4: Let x_0 be any arbitrary point of X, A(X)$\subseteq$ T(X) and B(X) $\subseteq$S(X) there exists $x_1,x_2\in$ X such that $Ax_0=Tx_1$ and $Bx_1=Sx_2$.
Inductively we construct a sequence $<x_n>$ and $<y_n>$ in X such that $y_{2n}=Ax_{2n}= Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+1} = Sx_{2n+2}$ for n ≥ 0.
By putting x= x_{2n} , y= x_{2n+1} in the inequality 2.1.3

$$M(y_{2n}y_{2n+1},kt)$$
$$=[M(Ax_{2n},Bx_{2n+1},kt)]$$
$$\geq \min\{M(Sx_{2n},Tx_{2n+1},t), M(Ax_{2n},Sx_{2n},t), M(Bx_{2n+1},Tx_{2n+1},t), M(Bx_{2n+1},Sx_{2n},2t), M(Ax_{2n},Tx_{2n+1},t)\}$$
$$\geq \min\{M(y_{2n-1},y_{2n},t), M(y_{2n},y_{2n-1},t), M(y_{2n+1},y_{2n},t), M(y_{2n+1},y_{2n-1},2t), M(y_{2n},y_{2n},t)\}$$
$$\geq \min\{M(y_{2n-1},y_{2n},t), M(y_{2n+1},y_{2n},t), M(y_{2n+1},y_{2n-1},2t), 1\}$$
$$\geq \min\{M(y_{2n-1},y_{2n},t), M(y_{2n+1},y_{2n},t), M(y_{2n+1},y_{2n},t), M(y_{2n-1},y_{2n},t), 1\}$$
$$\geq \min\{M(y_{2n-1},y_{2n},t), M(y_{2n+1},y_{2n},t), 1\}$$
$$\geq M(y_{2n-1},y_{2n},t)$$
$$In\ general\ M(y_n y_{n+1},kt) \geq M(y_{n-1},y_n,t) \qquad ....(1)$$

to prove $\{Y_N\}$ is cauchy sequence we prove $M(Y_N, Y_M,T)>1-\in$ ………..(2) is true for all $N\geq N_0$ and for every M$\in$N.

Here we use induction method, From (1) we have

$$\left[M(y_n, y_{n+1}, t)\right] \geq M(y_{n-1}, y_n, t/k) \geq M(y_{n-2}, y_{n-1}, t/k^2) \geq .... \geq M(y_1, y_2, t/k^n)$$

$$M(y_n, y_{n+1}, t) \rightarrow 1 \quad as\ n \rightarrow \infty$$

i.e. for each t>0 and each ε>0 we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1-\varepsilon$ for all $n > n_0$.
For m,n∈N, suppose m≥n, then we have that
$M(y_n, y_m, t) \geq$

$$\min\left\{M(y_n, y_{n+1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) * ...... * M(y_{m-1}, y_m, t/m-n)\right\}$$

$$> (1-\varepsilon)*(1-\varepsilon)*...(1-\varepsilon) \geq (1-\varepsilon) \quad (m-n)\, times$$

$$\geq 1-\varepsilon$$

thus $\{y_n\}$ is a Cauchy sequence. By completeness of (X, M, *), {yn} converges to some point z in X.Also its subsequences converges to some point z∈X
i.e $Ax_{2n}, \rightarrow z$ $Tx_{2n+1} \rightarrow z$, $Bx_{2n+1} \rightarrow z$, $Tx_{2n+1} \rightarrow z$

Proof of main Theorem 2.1:

From the lemma we $Ax_{2n} \rightarrow z$ $Tx_{2n+1} \rightarrow z$, $Bx_{2n+1} \rightarrow z$, $Tx_{2n+1} \rightarrow z$
Since the condition A(X) ⊆ T(X) implies there exists u ∈ X such that T u=z
The pair (A,S) is reciprocally continuous implies $ASx_{2n}, \rightarrow Az$ $SAx_{2n}, \rightarrow Sz$ as n→∞.
And also from definition of the pair (A,S) is compatible, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all t>0 when ever $Ax_n, \rightarrow z$ $Sx_n, \rightarrow z$ as n→∞ .
Which implies $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n$ gives Az=Sz.
To prove Bu=z
Put x= x_{2n} , y=u. in the inequality 2.1.3

$$\left[M(Ax_{2n}, Bu, kt)\right] \geq \min\left\{M(Sx_{2n}, Tu, t), M(Ax_{2n}, Sx_{2n}, t), M(Bu, Tu, t), M(Bu, Sx_{2n}, 2t), M(Ax_{2n}, Tu, t)\right\}$$

$$\left[M(z, Bu, kt)\right] \geq \min\left\{M(z, z, t), M(z, z, t), M(Bu, z, t), M(Bu, z, 2t), M(z, z, t)\right\}$$

$$\left[M(z, Bu, kt)\right] \geq M(Bu, z, t)$$

$$Bu = z$$

Since (B, T) is weakly compatible , $BTu = TBu \Rightarrow Bz = Tz$.

To prove Az=z
Put x=z, y= x_{2n+1} in the inequality 2.1.3

$$\left[M(Az, Bx_{2n+1}, kt)\right] \geq \min\left\{M(Sz, Tx_{2n+1}, t), M(Az, Sz, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Bx_{2n+1}, Sz, 2t), M(Az, Tx_{2n+1}, t)\right\}$$

$$\left[M(Az, z, kt)\right] \geq \min\left\{M(Az, z, t), M(Az, Az, t), M(z, z, t), M(z, Az, 2t), M(Az, z, t)\right\}$$

$$\left[M(Az, z, kt)\right] \geq M(Az, z, t)$$

$$Az = z$$

$$\therefore Az = Sz = z$$

To prove Bz=z

Put x=z,y= z in the inequality 2.1.3

$$\left[M(Az,Bz,kt)\right] \geq \min\left\{M(Sz,Tz,t), M(Az,Sz,t), M(Bz,Tz,t), M(Bz,Sz,2t), M(Az,Tz,t)\right\}$$

$$\left[M(z,Bz,kt)\right] \geq \min\left\{M(z,Bz,t), M(z,z,t), M(Bz,Bz,t), M(Bz,z,2t), M(z,Bz,t)\right\}$$

$$\left[M(z,Bz,kt)\right] \geq M(z,Bz,t)$$

$$Bz = z$$

$$\therefore Bz = Tz = z$$

since Az=Bz=Sz=Tz=z,we get z in a common fixed point of A,B,S and T.

Uniqueness:

Let w be a another common fixed point of above self maps then we have Aw=Bw=Sw=Tw=w

$$\left[M(Az,Bw,kt)\right] \geq \min\left\{M(Sz,Tw,t), M(Az,Sw,t), M(Bw,Tw,t), M(Bw,Sz,2t), M(Az,Tw,t)\right\}$$

$$\left[M(z,w,kt)\right] \geq M(z,w,t)$$

Hence $z = w$ this completes the proof

2.3 Example: Let X= [0,1], $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ where d(x,y)=$|x-y|$

$$Ax = Bx = Sx = \begin{cases} \frac{1}{8} & if\ 0 \leq x \leq \frac{1}{8} \\ \frac{1}{6} & if\ \frac{1}{8} < x \leq 1 \end{cases} \qquad Tx = \begin{cases} \frac{1}{4} - x & if\ 0 \leq x \leq \frac{1}{8} \\ \frac{1}{4} & if\ \frac{1}{8} < x \leq 1 \end{cases}$$

then A(X) =B(X)= S(X) = $\left\{\frac{1}{8}, \frac{1}{6}\right\}$ while T(X) = $\left\{\frac{1}{4} \cup \left[\frac{1}{4}, \frac{1}{8}\right]\right\}$ so that the conditions A(X) $\subseteq$ T(X) and B(X) $\subseteq$ S(X) are satisfied .

From the example given above, clearly the pairs (A,S) is reciprocally continuous and compatible and (B,T) are weakly compatible as they commute at coincident point 1/8 .

Let a sequence $x_n = \left(\frac{1}{6} + \frac{1}{n}\right)$ for n≥1, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1/8$,

$\lim_{n\to\infty} ASx_n = 1/8$, $\lim_{n\to\infty} SAx_n = 1/8$ So that $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$

and $\lim_{n\to\infty} BTx_n = 1/6$, $\lim_{n\to\infty} TBx_n = 1/8$ So that $\lim_{n\to\infty} M(BTx_n, TBx_n, t) \neq 1$.

Also note that none of the mappings are continuous.

Clearly 1/8 is the unique common fixed point of A, B, S and T.

REFERENCES

1. Balasubramaniam P, Murali Sankar S & Pant R P, “Common fixed point of four mappings in fuzzy metric space”, *J. Fuzzy Math.* 10(2), pp.379-384, (2002).
2. Bijendra Singh and Chauhan, “Common fixed points of compatible mapps in fuzzy metric spaces”, *Fuzzy sets and Systems*,115,pp.471-475, (2005).
3. George A, Veeramani P, “On some results in fuzzy metric spaces”, Fuzzy Sets and Systems, 64, pp.395-399, (1994).
4. Grabiec M,” Fixed points in fuzzy metric spaces”, *Fuzzy Sets and Systems*, 27, pp.385-389, (1998).
5. Jungck G, “Compatible mappings and common fixed points (2)”, Internet. *J. Math & Math. Sci*., pp. 285-389, (1988).
6. Kutukcu S, Sharma S & Tokgoz H A, “A fixed point theorem in fuzzy metric spaces”, *Int. J. Math. Analysis,* 1(18) ,pp. 861-872, (2007).
7. Mishra S N, Sharma N & Singh S L, “Common fixed points of mappings on fuzzy metric spaces”, *Internet J. Math & Math. Sci*, 17, pp.253-258, (1994).
8. Pant R P & Jha K, Aremarkon, “Common fixed points of four mappings in a fuzzy metric space”, *J. Fuzzy Math*.12 (2) ,4, pp.433-437, (2004).
9. Sharma S,” Common fixed point theorems in fuzzy metric spaces”, *Fuzzy Sets and Systems* 127, pp.345-352, (2002).
10. Sisodia K S, Rathore M S, & Deepak Singh ,” *A common fixed point theorem in fuzzy metric spaces*”, Vol.5, No.17, pp. 819-826, (2011).
11. Zadeh L A, “Fuzzy Sets”, *Information and Control*, 8, pp.338-353, (1965).