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Derivations of e – Commutative BF_1 – algebra

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ABSTRACT

In this paper, the notion of left–right (respectively, right–left) derivation of an e–commutative BF_1–algebra has been introduced. The concepts of regular derivations and identity derivations of an e–commutative BF_1 – algebra have been studied and several related properties were investigated.

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1. INTRODUCTION

In 2007, Walendziak[12] introduced the classes of *BF, BF_1, BF_2 – algebras*, which are generalizations of *B – algebra*[7]. He defined *BF – algebra* as an algebraic structure *$(A,*, 0)$ of type (2,0)*, satisfying *(I)* $x * x = 0$, *(II)* $x * 0 = x$, *(BF)* $0 * (x * y) = y * x$, for all *x, y* in *A*. Kim and Kim[6] introduced BG–algebra $(A,*, 0)$ of type *(2,0)*, satisfying *(I)*, *(II)* and *(BG)* $(x * y) * (0 * y) = x$. Walendziak further extended *BF–algebra* to *BF_1–algebra*, by including the property *(BG)*. Motivated by the concepts of derivations of BCI–algebras by Jun and Xin[13], some results on derivations of BCI–algebras by Hamza *et al*.[1], derivations of B–algebras by Nora O. Al-Shehrie[8] and Derivations on QS–algebras by Mostafa *et al*.[9], authors introduced the concepts of *(left-right)–derivation, (right-left)–derivation, regular and identity derivations* of an *e–commutative BF_1–algebra* and investigated several related properties, which may be a contribution to the theories of *propositional calculi* [2-5,10].

2. NOTATIONS

Throughout this article, authors used the notations *D*: $e * (e * x) = x$, *E*: $x * (e * y) = y * (e * x)$, *F*: $y * (y * x) = x$, *G*: $(e * x) * (e * y) = e * (x * y) = y * x$, $(BF_1)^e$: *X* is an *e –*

commutative BF_1–algebra, *Δ: derivation*, *(l, r)–Δ*: *(l, r)–derivation* and *(r, l)–Δ*: *(r, l)–derivation*, $\forall$ *x, y, z* $\in X$ and for any fixed $e \in X$.

3. PRELIMINARIES

Definition 3.1. A BF_1– *algebra* $(X,*, e)$ is said to be an $(BF_1)^e$**,** if $x * (e * y) = y * (e * x)$, for all $x, y \in X$.
Definition 3.2. [10, Proposition 3.7] If $(X,*, e)$ is an $(BF_1)^e$ then $(e * x) * y = (e * y) * x$, for all $x, y \in X$**.**
Proposition 3.3. [11, Proposition 3.2] Let $(X,*, e)$, for any fixed $e \in X$ be a BF – Algebra. Then X is an $(BF_1)^e$ if and only if $(e * x) * (e * y) = y * x = e * (x * y)$, for all $x, y \in X$.
Corollary 3.4. [11, Corollary 3.3] Let $(X,*, e)$, for any fixed $e \in X$ be a BF_1– *algebra*. Then X is an $(BF_1)^e$ if and only if $(e * x) * (e * y) = y * x = e * (x * y)$, for all $x, y \in X$.
Corollary 3.5. [11, Corollary 3.4] Let $(X,*, e)$, for any fixed $e \in X$ be a BF_2– *algebra*. Then X is an $(BF_1)^e$ if and only if $(e * x)*(e * y) = y * x = e * (x * y)$, for all $x, y \in X$.
Definition 3.6. Let $(X,*, e)$ be an $(BF_1)^e$. Then the partial order " $\leq$ " is defined as $x \leq y$ if and only if $x * y = e$, $\forall x, y \in X$ and $x \wedge y$ is defined as, $x \wedge y = y * (y * x)$, for all $x, y \in X$.
Definition 3.7. Let $(X,*, e)$ be an $(BF_1)^e$. A self map $Δ: X \to X$ is said to be $(l, r) – Δ$ of X, if it satisfies the identity $Δ (x * y) = (Δ (x) * y) \wedge (x * Δ (y))$, for all $x, y \in X$.
Definition 3.8. Let $(X,*, e)$ be an $(BF_1)^e$. A self map $Δ: X \to X$ is said to be $(r, l) – Δ$ of X if, it satisfies the identity $Δ (x * y) = (x * Δ (y)) \wedge (Δ (x) * y)$, for all $x, y \in X$.
Definition 3.9. Let $(X,*, e)$ be an $(BF_1)^e$. A self map $Δ: X \to X$ is said to be a derivation of X if, it is both $(l, r) – Δ$ and $(r, l) – Δ$ on X.
Proposition 3.10. [6, Lemma 2.4] Cancellation Laws holds well in BG – algebra.
Proposition 3.11. [10, Lemma 3.1] Cancellation Laws holds well in an $(BF_1)^e$.

Example 3.12. Let $X = \{e, a, b, c\}$ and $*$ be the binary operation defined on X as shown below.

*	e	a	b	c
e	e	a	c	b
a	a	e	b	c
b	b	c	e	a
c	c	b	a	e

Define a map $Δ: X \to X$ such that

$$Δ(x) = \begin{cases} e, & if \quad x = a \\ a, & if \quad x = e \\ b, & if \quad x = c \\ c, & if \quad x = b \end{cases}.$$

Then one can easily verify that $Δ$ is a derivation of X.
Note 3.13. From example 3.12, it is clear that, (i) $Δ$ is not a regular derivation of X as $Δ (e) \neq e$ and (ii) $Δ$ is not an identity derivation of X as $Δ (x) \neq x$, for all $x \in X$.

4. RESULTS ON *(l, r)* and *(r, l)* – derivations

Proposition 4.1. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ is a $(l, r) - \Delta$ of X. Then $\Delta (x * y) = \Delta (x) * y$, for all $x, y \in X$.

Proof: Since Δ is a $(l, r) - \Delta$ of X then $\Delta (x * y) = (\Delta (x) * y) \wedge (x * \Delta (y)) = (x * \Delta (y)) * ((x * \Delta (y)) * (\Delta (x) * y)) = e * (((x * \Delta (y)) * (\Delta (x) * y)) * (x * \Delta (y))) = (e * ((x * \Delta (y)) * (\Delta (x) * y))) * (e * (x * \Delta (y)) = ((\Delta (x) * y) * (x * \Delta (y)) * (e * (x * \Delta (y))) = \Delta (x) * y$

$\therefore \Delta (x * y) = \Delta (x) * y, \ \forall x, y \in X$.

Proposition 4.2. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ is a $(r, l) - \Delta$ of X. Then $\Delta (x * y) = x * \Delta (y)$, for all $x, y \in X$.

Proof: Proof is similar to the proof of proposition 4.1.

Theorem 4.3. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ is a derivation of X. Then $\Delta (x * y) = \Delta (x) * y = x * \Delta (y)$, for all $x, y \in X$.

Proof: The theorem can be proved by combining proposition 4.1 and proposition 4.2.

Definition 4.4. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then the composition mapping of Δ_1 and Δ_2 is denoted by $\Delta_1 \, o \, \Delta_2$ and is defined as, $(\Delta_1 \, o \, \Delta_2)(x) = \Delta_1 (\Delta_2 (x))$, for all $x \in X$.

Proposition 4.5. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then the composition mapping $\Delta_1 \, o \, \Delta_2$ is also a $(l, r) - \Delta$ of X.

Proof: Since Δ_1, Δ_2 be the two $(l, r) - \Delta s$ of X then $\Delta_1 (x * y) = \Delta_1 (x) * y, \ \forall x, y \in X$ and $\Delta_2 (x * y) = \Delta_2 (x) * y, \ \forall x, y \in X$.

Consider, $(\Delta_1 \, o \, \Delta_2) (x * y) = \Delta_1 (\Delta_2 (x * y)) = \Delta_1 (\Delta_2 (x) * y) = \Delta_1 (\Delta_2 (x)) * y = (\Delta_1 \, o \, \Delta_2) (x) * y$. Hence, $(\Delta_1 \, o \, \Delta_2) (x * y) = (\Delta_1 \, o \, \Delta_2) (x) * y, \ \forall x, y \in X$.

Proposition 4.6. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two $(r, l) - \Delta s$ of X. Then the composition mapping $\Delta_1 \, o \, \Delta_2$ is also a $(r, l) - \Delta$ on X.

Proof: Proof is similar to the proof of proposition 4.5.

Theorem 4.7. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then the composition mapping $\Delta_1 \, o \, \Delta_2$ is also a derivation of X.

Proof: The theorem can be proved by combining proposition 4.5 and proposition 4.6.

Theorem 4.8. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then $(\Delta_1 \, o \, \Delta_2) (x * y) = (\Delta_2 \, o \, \Delta_1) (x * y)$, for all $x, y \in X$.

Proof: Since Δ_1, Δ_2 be the two derivations of X, then Δ_1, Δ_2 are both (l, r) – and $(r, l) - \Delta s$ on X. Consider, $(\Delta_1 \, o \, \Delta_2) (x * y) = \Delta_1 (\Delta_2 (x * y)) = \Delta_1 (\Delta_2 (x) * y) = \Delta_2 (x) * \Delta_1 (y) = \Delta_2 (x * \Delta_1 (y)) = \Delta_2 (\Delta_1 (x * y)) = (\Delta_2 \, o \, \Delta_1) (x * y)$. Hence, $(\Delta_1 \, o \, \Delta_2) (x * y) = (\Delta_2 \, o \, \Delta_1) (x * y)$. $\therefore (\Delta_1 o \, \Delta_2) (x * y) = (\Delta_2 \, o \, \Delta_1) (x * y), \ \forall x, y \in X$.

Corollary 4.9. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then $(\Delta_1 \, o \, \Delta_2) (x) = (\Delta_2 \, o \, \Delta_1)(x)$, for all $x, y \in X$.

Proof: Taking $y = e$ in theorem 4.8, the result can be obtained.

Definition 4.10. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Define $\Delta_1 * \Delta_2 : X \to X$ such that $(\Delta_1 * \Delta_2)(x) = \Delta_1(x) * \Delta_2(x)$, for all $x \in X$.

Theorem 4.11. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be two derivations of X. Then $\Delta_1 * \Delta_2 = \Delta_2 * \Delta_1$.

Proof: Consider, $(\Delta_1 \, o \, \Delta_2)(x * y) = \Delta_1(x * \Delta_2(y)) = \Delta_1(x) * \Delta_2(y)$ and similarly $(\Delta_1 \, o \, \Delta_2)(x * y) = \Delta_1(\Delta_2(x * y) = \Delta_1(\Delta_2(x) * y) = \Delta_2(x) * \Delta_1(y)$.

i.e. $\Delta_1(x) * \Delta_2(y) = \Delta_2(x) * \Delta_1(y)$, replacing y by x, $\Delta_1(x) * \Delta_2(x) = \Delta_2(x) * \Delta_1(x)$.

i.e. $(\Delta_1 * \Delta_2)(x) = (\Delta_2 * \Delta_1)(x)$. Hence, $\forall x \in X, \Delta_1 * \Delta_2 = \Delta_2 * \Delta_1$.

Definition 4.12. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Define $\Delta^2 : X \to X$ such that $\Delta^2(x) = (\Delta \, o \, \Delta)(x) = \Delta(\Delta(x))$, for all $x \in X$.

Definition 4.13. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Define $\Delta^n : X \to X$ such that $\Delta^n(x) = \Delta^{n-1}(\Delta(x))$, for all $x \in X$ and $n \in Z^+$.

Proposition 4.14. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the $(l, r) - \Delta$ of X. Then $\Delta^2(x * y) = \Delta^2(x) * y, \ \forall x, y \in X$.

Proof: Consider, $\Delta^2(x * y) = (\Delta \, o \, \Delta)(x * y) = \Delta(\Delta(x * y)) = \Delta(\Delta(x) * y)) = \Delta(\Delta(x)) * y$
$= (\Delta \, o \, \Delta)(x) * y = \Delta^2(x) * y$.

$\therefore \Delta^2(x * y) = \Delta^2(x) * y, \ \forall x, y \in X$.

Proposition 4.15. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the $(r, l) - \Delta$ of X. Then $\Delta^2(x * y) = x * \Delta^2(y)$, for all $x, y \in X$.

Proof: Proof is similar to the proof of theorem 4.14.

Theorem 4.16. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $\Delta^2(x * y) = \Delta(x) * \Delta(y)$, for all $x, y \in X$.

Proof: Consider, $\Delta^2(x * y) = (\Delta \, o \, \Delta)(x * y) = \Delta(\Delta(x * y)) = \Delta(x * \Delta(y)) = \Delta(x) * \Delta(y)$

$\therefore \Delta^2(x * y) = \Delta(x) * \Delta(y), \ \forall x, y \in X$.

Proposition 4.17. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the self map of X such that $\Delta(x) = e * (e * x)$, for all $x \in X$. Then Δ is a $(l, r) - \Delta$ of X.

Proof: Consider, $(\Delta(x) * y) \wedge (x * \Delta(y)) = (x * \Delta(y)) * ((x * \Delta(y)) * (\Delta(x) * y))$

$= (x * \Delta(y)) * (e * ((\Delta(x) * y) * (x * \Delta(y))))$ (BF)

$= ((\Delta(x) * y) * (x * \Delta(y)) * (e * (x * \Delta(y)))$ (E)

$= \Delta(x) * y$ (BG)

$= (e * (e * x)) * y = x * y$ (D)

$= e * (e * (x * y))$ (D) $= \Delta(x * y)$

Hence, $\Delta(x * y) = (\Delta(x) * y) \wedge (x * \Delta(y)), \ \forall x, y \in X$.

$\therefore \Delta$ is a $(l, r) - \Delta$ of X.

Proposition 4.18. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the self map of X such that $\Delta(x) = e * (e * x)$, for all $x \in X$. Then Δ is a $(r, l) - \Delta$ of X.

Proof: Proof is similar to the proof of theorem 4.17.

Theorem 4.19. Let Δ be the self map of an $(BF_1)^e\ (X,*,e)$ such that $\Delta(x) = e*(e*x)$, for all $x\in X$. Then Δ is a derivation of X.

Proof: The theorem can be proved by combining the proposition 4.17 and proposition 4.18.

Proposition 4.20. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the $(r,l)-\Delta$ of X. Then $\Delta(x) = e*(e*\Delta(x))$, for all $x\in X$.

Proof: Consider, $\Delta(x) = \Delta(e*(e*x)) = e*\Delta(e*x) = e*(e*\Delta(x))$.

$\therefore \Delta(x) = e*(e*\Delta(x)),\ \forall x\in X$.

Proposition 4.21. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the $(r,l)-\Delta$ of X. Then $x*(e*\Delta(y)) = y*(e*\Delta(x))$, for all $x, y\in X$.

Proof: Consider, $x*(e*\Delta(y)) = x*\Delta(e*y) = \Delta(x*(e*y)) = \Delta(y*(e*x) = y*\Delta(e*x) = y*(e*\Delta(x))$.

$\therefore x*(e*\Delta(y)) = y*(e*\Delta(x)),\ \forall x, y\in X$.

Proposition 4.22. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the $(l,r)-\Delta$ of X. Then $x*(e*\Delta(y)) = y*(e*\Delta(x))$, for all $x, y\in X$.

Proof: Proof is similar to the proof of proposition 4.21.

Theorem 4.23. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $x*(e*\Delta(y)) = y*(e*\Delta(x))$, for all $x, y\in X$.

Proof: The theorem can be proved by combining proposition 4.21 and proposition 4.22.

Proposition 4.24. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the $(r,l)-\Delta$ of X. Then $(e*x)*\Delta(y) = (e*y)*\Delta(x)$, for all $x, y\in X$.

Proof: Consider, $(e*x)*\Delta(y) = \Delta((e*x)*y) = \Delta((e*y)*x), = (e*y)*\Delta(x)$.

$\therefore (e*x)*\Delta(y) = (e*y)*\Delta(x), \forall x, y\in X$.

Proposition 4.25. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the $(l,r)-\Delta$ of X. Then $(\Delta(e)*x)*y = (\Delta(e)*y)*x$, for all $x, y\in X$.

Proof: Consider, $(\Delta(e)*x)*y = \Delta(e*x)*y = \Delta((e*x)*y) = \Delta((e*y)*x)$,

$$= \Delta(e*y)*x = (\Delta(e)*y)*x.$$

Hence, $(\Delta(e)*x)*y = (\Delta(e)*y)*x, \forall x, y\in X$.

Proposition 4.26. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(e*\Delta(x))*y = (e*\Delta(y))*x$, for all $x, y\in X$.

Proof: Similar proof as done in proposition 4.25.

Definition 4.27. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be the two derivations of X. Then define $(\Delta_1\wedge\Delta_2)(x) = \Delta_1(x)\wedge\Delta_2(x)$, for all $x\in X$.

Proposition 4.28. Let $(X,*,e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be the two $(l,r)-\Delta s$ of X. Then $\Delta_1\wedge\Delta_2$ is also a $(l,r)-\Delta$ of X.

Proof: To prove that $(\Delta_1\wedge\Delta_2)(x*y) = (\Delta_1\wedge\Delta_2)(x)*y,\ \forall x, y\in X$. Consider, $(\Delta_1\wedge\Delta_2)(x*y) = \Delta_1(x*y)\wedge\Delta_2(x*y) = (\Delta_1(x)*y)\wedge(\Delta_2(x)*y)$

$= (\Delta_2(x)*y)*((\Delta_2(x)*y)*(\Delta_1(x)*y))$ (F)

$= (\Delta_2(x)*y)*(e*((\Delta_1(x)*y)*(\Delta_2(x)*y)))$ (BF)

$= ((\Delta_1(x)*y)*(\Delta_2(x)*y))*(e*(\Delta_2(x)*y))$ (E)

$= ((\Delta_1(x) * y) * (\Delta_2(x) * y)) * (e * (\Delta_2(x) * y))$ (BF)
$= \Delta_1(x) * y$ (BG)
$= (\Delta_2(x) * (\Delta_2(x) * \Delta_1(x))) * y$ (F)
$= (\Delta_1(x) \wedge \Delta_2(x)) * y = (\Delta_1 \wedge \Delta_2)(x) * y.$
$\therefore (\Delta_1 \wedge \Delta_2)(x * y) = (\Delta_1 \wedge \Delta_2)(x) * y, \ \forall x, y \in X.$

Proposition 4.29. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ_1, Δ_2 be the two $(r, l) - \Delta s$ of X. Then $\Delta_1 \wedge \Delta_2$ is also a $(r, l) - \Delta$ of X.

Proof: Proof is similar to the proof of proposition 4.28.

Theorem 4.30. If Δ_1, Δ_2 be the two derivations of an $(BF_1)^e$ $(X,*, e)$.Then $\Delta_1 \wedge \Delta_2$ is also a derivation of X.

Proof: The theorem can be proved by combining proposition 4.28 and proposition 4.29.

Theorem 4.31. If $(X,*, e)$ be an $(BF_1)^e$ such that and $e * \Delta(x) = \Delta(x)$, for all $x \in X$ then $(e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x)$, for all $x, y \in X$.

Proof: Consider, $(e * \Delta(x)) * \Delta(y) = (e * \Delta(x)) * (e * \Delta(y)) = (\Delta \circ \Delta)((e * x) * (e * y)) = (\Delta \circ \Delta)(e * (x * y)) = e * (\Delta(x) * \Delta(y)) = \Delta(y) * \Delta(x) = (\Delta(y) * e) * \Delta(x) = (e * \Delta(y)) * \Delta(x)$
$\therefore (e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x), \forall x, y \in X.$

Theorem 4.32. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X such that $\Delta(z) = e * \Delta(y)$, for all $y, z \in X$. Then $\forall x, y, z \in X$, $((\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * ((\Delta(z) * (e * \Delta(y))$.

Proof: Consider, $((\Delta(x) * \Delta(y)) * \Delta(z) = ((\Delta(x) * \Delta(y)) * (e * \Delta(y))$
$= ((\Delta \circ \Delta) \circ \Delta)((x * y) * (e * y) = ((\Delta \circ \Delta) \circ \Delta)(x * e) = ((\Delta \circ \Delta) \circ \Delta)(x * ((e * y) * (e * y)))$
$= \Delta(\Delta(x) * ((e * \Delta(y)) * (e * y))) = \Delta(\Delta(x) * (\Delta(z) * (e * y))) = \Delta(x) * ((\Delta(z) * (e * \Delta(y)).$
$\therefore ((\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * ((\Delta(z) * (e * \Delta(y)), \ \forall x, y, z \in X.$

Proposition 4.33. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y))$, for all $x, y, z \in X$.

Proof: Consider, $(e * \Delta(x)) * (e * \Delta(y)) = (\Delta \circ \Delta)((e * x) * (e * y)) = (\Delta \circ \Delta)(y * (e * (e * x))) = (\Delta \circ \Delta)(y * x)$ (i)
$= \Delta(y) * \Delta(x)$
From (i), $(e * \Delta(x)) * (e * \Delta(y)) = (\Delta \circ \Delta)(e * (x * y)) = e * (\Delta(x) * \Delta(y))$ (ii)
From relations (i) and (ii), $(e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y)), \ \forall x, y \in X.$

Proposition 4.34. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $\Delta(x) * (e * \Delta(y)) = \Delta(y) * (e * \Delta(x))$, for all $x, y \in X$.

Proof: Consider, $\Delta(x) * (e * \Delta(y)) = (\Delta \circ \Delta)(x * (e * y)) = (\Delta \circ \Delta)(y * (e * x)) = \Delta(y) * (e * \Delta(x)).$
$\therefore \Delta(x) * (e * \Delta(y)) = \Delta(y) * (e * \Delta(x)), \ \forall x, y \in X.$

Proposition 4.35. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y))$ if and only if $\Delta(x) * (e * \Delta(y)) = \Delta(y) * (e * \Delta(x)), \forall x \in X$.

Proof: Suppose that $(e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y)), \ \forall x, y \in X$ holds good. Consider, $\Delta(x) * (e * \Delta(y)) = e * ((e * \Delta(y)) * \Delta(x)) = (e * (e * \Delta(y)) * (e * \Delta(x))$
$= \Delta(y) * (e * \Delta(x)).$

$\therefore \Delta(x) * (e * \Delta(y)) = \Delta(y) * (e * \Delta(x)),\ \forall x, y \in X$. Conversely, suppose that $\Delta(x) * (e * \Delta(y)) = \Delta(y) * (e * \Delta(x)),\ \forall x, y \in X$ holds good.

Consider, $(e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * (e * (e * \Delta(x)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y))$

$\therefore (e * \Delta(x)) * (e * \Delta(y)) = \Delta(y) * \Delta(x) = e * (\Delta(x) * \Delta(y)),\ \forall x, y \in X$.

Proposition 4.36. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x)$, for all $x, y \in X$.

Proof: Since $(X, *, e)$ is an $(BF_1)^e$ then, $x * (e * y) = y * (e * x),\ \forall x, y \in X$.

Consider, $(e * \Delta(x)) * \Delta(y) = (\Delta \circ \Delta)((e * x) * y) = (\Delta \circ \Delta)((e * x) * (e * (e * y)) = (\Delta \circ \Delta)(e * y) * (e * (e * x)) = (\Delta \circ \Delta)((e * y) * x) = (e * \Delta(y)) * \Delta(x)$

$\therefore (e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x),\ \forall x, y \in X$.

Proposition 4.37. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x)$, for all $x, y \in X$.

Proof: Consider, $(e * \Delta(x)) * \Delta(y) = (\Delta \circ \Delta)((e * x) * y) = (\Delta \circ \Delta)(e * (y * (e * x)) = (\Delta \circ \Delta)((e * y) * (e * (e * x))) = (\Delta \circ \Delta)((e * y) * x) = (e * \Delta(y)) * \Delta(x))$

$\therefore (e * \Delta(x)) * \Delta(y) = (e * \Delta(y)) * \Delta(x),\ \forall x, y \in X$.

Theorem 4.38. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X such that $\Delta(z) = (\Delta(x) * \Delta(y))$, for all $x, y, z \in X$. Then $(\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * (\Delta(z) * (e * \Delta(y))$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta(x) * \Delta(y)) * \Delta(z) = (\Delta(x) * \Delta(y)) * (\Delta(x) * \Delta(y)) = (\Delta \circ \Delta \circ \Delta \circ \Delta)(e) = (\Delta \circ \Delta \circ \Delta \circ \Delta)(x * x) = (\Delta \circ \Delta \circ \Delta \circ \Delta)(x * ((x * y) * (e * y))) = \Delta(x) * ((\Delta(x) * \Delta(y)) * ((e * \Delta(y)))$

$\therefore (\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * (\Delta(z) * (e * \Delta(y)),\ \forall x, y, z \in X$.

Theorem 4.39. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X such that $\Delta(z) = e * \Delta(y)$, for all $y, z \in X$. Then $(\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * (\Delta(z) * (e * \Delta(y)),\ \forall x, y, z \in X$.

Proof: Consider, $(\Delta(x) * \Delta(y)) * \Delta(z) = (\Delta(x) * \Delta(y)) * (e * \Delta(y)) = (\Delta \circ \Delta \circ \Delta)((x * y) * (e * y)) = (\Delta \circ \Delta \circ \Delta)(x) = (\Delta \circ \Delta \circ \Delta)(x * e) = (\Delta \circ \Delta \circ \Delta)(x * ((e * y) * (e * y))) = \Delta(x) * ((e * \Delta(y)) * (e * \Delta(y)) = \Delta(x) * (\Delta(z) * (e * \Delta(y))$.

$\therefore (\Delta(x) * \Delta(y)) * \Delta(z) = \Delta(x) * (\Delta(z) * (e * \Delta(y)),\ \forall x, y, z \in X$.

Theorem 4.40. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $\Delta(z) = \Delta(x) * \Delta(y)$, for all $x, y, z \in X$. Then $(\Delta(x) * \Delta(y)) * \Delta(z) = (e * \Delta(z)) * (\Delta(y) * \Delta(x))$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta(x) * \Delta(y)) * \Delta(z) = (\Delta(x) * \Delta(y)) * (\Delta(x) * \Delta(y))$
$= (\Delta \circ \Delta \circ \Delta \circ \Delta)(e) = (\Delta \circ \Delta \circ \Delta \circ \Delta)(y * x) * (y * x) = (\Delta(y) * \Delta(x)) * (\Delta(y) * \Delta(x))$
$= (e * (\Delta(x) * \Delta(y))) * (\Delta(y) * \Delta(x)) = (e * \Delta(z)) * (\Delta(y) * \Delta(x))$.

$\therefore (\Delta(x) * \Delta(y)) * \Delta(z) = (e * \Delta(z)) * (\Delta(y) * \Delta(x)),\ \forall x, y, z \in X$.

Theorem 4.41. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $\Delta(z) = e * \Delta(y)$, for all $y, z \in X$. Then $(\Delta(x) * \Delta(y)) * \Delta(z) = (e * \Delta(z)) * (\Delta(y) * \Delta(x))$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta(x) * \Delta(y)) * \Delta(z) = (\Delta(x) * \Delta(y)) * (e * \Delta(y)) = (\Delta \circ \Delta \circ \Delta)((x * y) * (e * y)) = (\Delta \circ \Delta \circ \Delta)(x) = (\Delta \circ \Delta \circ \Delta)(y * (y * x)) = (\Delta \circ \Delta \circ \Delta)(e * (e * y)) * (y * x)) = (e * (e * \Delta(y))) * (\Delta(y) * \Delta(x)) = (e * \Delta(z)) * (\Delta(y) * \Delta(x))$.

Hence, $(\Delta (x) * \Delta (y)) * \Delta (z) = (e * \Delta (z)) * (\Delta (y) * \Delta (x)), \forall x, y, z \in X$.

Theorem 4.42. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $(\Delta (x) * \Delta (y)) * \Delta (z) = (e * \Delta (z)) * (\Delta (y) * \Delta (x))$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta o \Delta o \Delta) ((x * y) * z) = (\Delta o \Delta o \Delta) ((x * y) * (e * (e * z) = (\Delta o \Delta o \Delta) ((e * z) * (e * (x * y)) = (\Delta o \Delta o \Delta) ((e * z) * (y * x)) = (e * \Delta (z)) * (\Delta (y) * \Delta (x))$
$\therefore (\Delta (x) * \Delta (y)) * \Delta (z) = (e * \Delta (z)) * (\Delta (y) * \Delta(x)), \forall x, y, z \in X$.

Theorem 4.43. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X . Then $(\Delta (x) * \Delta (y)) * \Delta (z) = (e * \Delta (z)) * (\Delta (y) * \Delta (x))$, for all $x, y, z \in X$.

Proof: Let $(X, *, e)$ is an $(BF_1)^e$ with (G).

Consider, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta o \Delta o \Delta) ((x * y) * z) = (\Delta o \Delta o \Delta) (e * (z * (x * y)) = (\Delta o \Delta o \Delta) ((e * z) * (e * (x * y)) = (\Delta o \Delta o \Delta) ((e * z) * (y * x))) = (e * \Delta (z)) * (\Delta (y) * \Delta (x))$
$\therefore (\Delta (x) * \Delta (y)) * \Delta (z) = (e * \Delta (z)) * (\Delta (y) * \Delta (x)), \forall x, y, z \in X$.

Theorem 4.44. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $\Delta (z) = e * \Delta (y), \forall y, z \in X$. Then $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y)$ for all $x, y, z \in X$.

Proof: Consider, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (y)) * (e * \Delta (y)) = (\Delta o \Delta o \Delta) ((x * y) * (e * y)) = (\Delta o \Delta o \Delta) (x) = (\Delta o \Delta o \Delta) (x * (e * y)) * (e * (e * y)) = (\Delta o \Delta o \Delta) (x * (e * y)) * y) = ((\Delta (x) * (e * \Delta (y)) * \Delta (y)) = (\Delta o \Delta o \Delta) ((\Delta (x) * \Delta (z)) * \Delta (y))$
$\therefore (\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y), \forall x, y, z \in X$.

Theorem 4.45. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $\Delta (z) = \Delta (x) * \Delta (y)$, $\forall x, y, z \in X$. Then $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y)$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (y)) * (\Delta (x) * \Delta (y)) = (\Delta o \Delta o \Delta o \Delta) (e) = (\Delta o \Delta o \Delta o \Delta) (y * y) = (\Delta o \Delta o \Delta o \Delta) (((y * x) * (e * x)) * y) = (\Delta o \Delta o \Delta o \Delta) ((x * (e * (y * x)) * y) = (\Delta o \Delta o \Delta o \Delta) ((x * (x * y)) * y) = (\Delta (x) * (\Delta (x) * \Delta (y))) * \Delta (y) = (\Delta (x) * \Delta (z)) * \Delta (y)$
$\therefore (\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y), \forall x, y, z \in X$.

Theorem 4.46. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $\Delta (z) = \Delta (x) * \Delta (y)$, for all $x, y, z \in X$ and (G). Then for all $x, y, z \in X$, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y)$.

Proof: Consider, $(\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (y)) * (\Delta (x) * \Delta (y))$
$= (\Delta o \Delta o \Delta o \Delta) (x * y) * (x * y) = (\Delta o \Delta o \Delta o \Delta) (e) = (\Delta o \Delta o \Delta o \Delta) (y * y)$
$= (\Delta o \Delta o \Delta o \Delta) (((y * x) * (e * x)) * y) = (\Delta o \Delta o \Delta o \Delta) (e * ((e * x) * (y * x))) * y)$
$= (\Delta o \Delta o \Delta o \Delta) ((e * (e * x)) * (e * (y * x))) * y = (\Delta o \Delta o \Delta o \Delta) (x * (x * y)) * y$
$= (\Delta (x) * (\Delta (x) * \Delta (y)) * \Delta (y) = (\Delta (x) * \Delta (z)) * \Delta (y)$
$\therefore (\Delta (x) * \Delta (y)) * \Delta (z) = (\Delta (x) * \Delta (z)) * \Delta (y), \forall x, y, z \in X$.

Theorem 4.47. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then $\Delta (x) * \Delta (y) = e * (\Delta (y) * \Delta (x))$, for all $x, y \in X$.

Proof: Consider, $\Delta (x) * \Delta (y) = (\Delta o \Delta) (x * y)) = (\Delta o \Delta) (e * (y * x)) = e * (\Delta (y) * \Delta (x))$
Hence, $\Delta (x) * \Delta (y) = e * (\Delta (y) * \Delta (x)), \forall x, y \in X$.

Theorem 4.48. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X with $(x * y) * z = x * (z * (e * y))$. Then $(\Delta (x) * \Delta (y)) * \Delta (z) = \Delta (x) * (\Delta (y)) * (e * \Delta (z))$, for all $x, y, z \in X$.

Proof: Proof is similar to the proof of theorem 4.47.

5. RESULTS ON REGULAR AND IDENTITY DERIVATIONS

Definition 5.1. Let $(X, *, e)$ be an $(BF_1)^e$**.** A self map $\Delta: X \to X$ is said to be a regular derivation of X, if $\Delta (e) = e, e \in X$.

Definition 5.2. Let $(X, *, e)$ be an $(BF_1)^e$. A self map $\Delta: X \to X$ is said to be an identity derivation of X, if $\Delta (x) = x$, for all $x \in X$.

Proposition 5.3. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then $\Delta (x) \leq x$, for all $x \in X$.

Proof: Since Δ is regular $(l, r) - \Delta$ on X then $\Delta (e) = e$ and $\Delta (x * y) = \Delta (x) * y$, for all $x, y \in X$. Consider, $e = \Delta (e) = \Delta (x * x) = \Delta (x) * x \Rightarrow \Delta (x) \leq x$, for all $x \in X$. (by definition 3.6).

Proposition 5.4. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then $x \leq \Delta (x)$, for all $x \in X$.

Proof: Proof is similar to the proof of proposition 5.3.

Theorem 5.5. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then Δ is an identity derivation of X.

Proof: Combining the proofs of proposition 5.3 and proposition 5.4, the theorem can be proved.

Theorem 5.6. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the derivation of X. Then Δ is identity derivation of X if and only if Δ is regular derivation of X.

Proof: Suppose that Δ is an identity derivation of $X \Rightarrow \Delta (x) = x \Rightarrow \Delta (x) * x = e \Rightarrow \Delta (x * x) = e \Rightarrow \Delta (e) = e \Rightarrow \Delta$ is regular derivation of X. Conversely, suppose that Δ is regular derivation of X. $\Rightarrow e = \Delta (e) = \Delta (x * x) = \Delta (x) * x \Rightarrow \Delta (x) \leq x, \forall x \in X$. Also, $e = \Delta (e) = \Delta (x * x) = x * \Delta (x) \Rightarrow x \leq \Delta (x), \forall x \in X$. Combining the above two inequalities, $\Delta (x) = x, \forall x \in X$. Hence**,** Δ is an identity derivation of X.

Proposition 5.7. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(l, r) - \Delta$ of X. Then $\Delta (x) = \Delta (x) \wedge x$, for all $x \in X$.

Proof: Consider, $\Delta (x) = \Delta (x * e) = (\Delta (x) * e) \wedge (x * \Delta (e)) = \Delta (x) \wedge (x * e) = \Delta (x) \wedge x$. $\therefore \Delta (x) = \Delta (x) \wedge x, \forall x \in X$.

Proposition 5.8. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then $\Delta (x) = x \wedge \Delta (x)$, for all $x \in X$.

Proof: Proof is similar to the proof of proposition 5.7.

Theorem 5.9. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then $\Delta (x) = \Delta (x) \wedge x = x \wedge \Delta (x)$, for all $x \in X$.

Proof: Combining proposition 5.7 and proposition 5.8, the theorem can be proved.

Proposition 5.10. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the $(l, r) - \Delta$ of X. Then (a) $\Delta (e) = \Delta (x) * x$, for all $x \in X$. (b) Δ is an injective mapping of X.
(c) If there exist $x \in X$ such that $\Delta (x) = x$, for all $x \in X$ then Δ is an identity derivation of X.
(d) If there exist $y \in X$ such that $\Delta (x) * y = e$ (or) $y * \Delta (x) = e$, for all $x \in X$ then Δ is a constant

derivation of X.

Proof: (a) Since Δ is $(l, r) - \Delta$ of X then $\Delta (e) = \Delta (x * x) = \Delta (x) * x, \forall x \in X$. (b) Let $x, y \in X$ such that $\Delta (x) = \Delta (y)$. From (a), $\Delta (x) * x = \Delta (e) = \Delta (y) * y \Rightarrow \Delta (x) * x = \Delta (y) * y \Rightarrow \Delta (x) * x = \Delta (x) * y, \Rightarrow x = y$. Hence, Δ is an injective mapping of X. (c) Let there exist an element $x \in X$ such that $\Delta (x) = x, \forall x \in X$. Consider, $y = x * (x * y), \forall x, y \in X \Rightarrow \Delta (y) = \Delta (x * (x * y)) = \Delta (x) * (x * y) = x * (x * y) = y, \Delta (y) = y, \forall y \in X$. Hence, Δ is an identity derivation of X. (d) Let there exist an element $y \in X$ such that $\Delta (x) * y = e, \forall x \in X$. Consider, $\Delta (x) * y = e \Rightarrow \Delta (x) * y = y * y$, using *RCL*, $\Delta (x) = y, \forall x \in X$. Hence Δ is a constant derivation of X. Similarly, if $y * \Delta (x) = e$, then $\Delta (x) = y, \forall x \in X$. Hence, Δ is a constant derivation of X.

Proposition 5.11. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the $(r, l) - \Delta$ of X. Then (a) $\Delta (e) = x * \Delta (x)$, for all $x \in X$. (b) Δ is an injective mapping of X. (c) If there exist $x \in X$ such that $\Delta (x) = x, \forall x \in X$ then Δ is an identity mapping. (d) If there exist $y \in X$ such that $\Delta (x) * y = e$ (or) $y * \Delta (x) = e, \forall x \in X$ then Δ is a constant mapping of X.

Proof: Proof is similar to the proof of proposition 5.10.

Corollary 5.12. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then $\Delta^2 (e) = e$, for any fixed $e \in X$.

Corollary 5.13. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then $\Delta^n (e) = e$, $n \in Z^+$.

Theorem 5.14. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then $\Delta^2 (x) = x$, for all $x \in X$.

Proof: Since Δ is regular $(r, l) - \Delta$ of X, then $\Delta (e) = e$ and $\Delta (x * y) = x * \Delta (y)$.

Consider, $\Delta^2 (x) = (\Delta \circ \Delta) (x) = \Delta (\Delta (x)) = \Delta (\Delta (x * e)) = \Delta (x * \Delta (e)) = \Delta (x * e) = x * \Delta (e) = x * e = x$. Hence, $\Delta^2 (x) = x, \forall x \in X$.

Corollary 5.15. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then $\Delta^n (x) = x$, $n \in Z^+$ and for all $x \in X$.

Proposition 5.16. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then Δ is an identity derivation of X.

Proof: Consider, $x * (e * \Delta (y)) = \Delta (y) * (e * x)$, (by E) $= \Delta (y * (e * x)) = \Delta (x * (e * y)) = \Delta (x) * (e * y) = y * (e * \Delta(x))$, taking $x=e$, $\Delta (y) = y$.

Hence, $\Delta (y) = y, \forall y \in X$. As the element y is arbitrary, $\Delta (x) = x, \forall x \in X$.

$\therefore$ Δ is an identity derivation of X.

Proposition 5.17. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular $(l, r) - \Delta$ of X. Then Δ is an identity derivation of X.

Proof: Proof is similar to the proof of proposition 5.16.

Theorem 5.18. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then Δ is an identity derivation on X.

Proof: The theorem can be proved by combining proposition 5.16 and proposition 5.17.

Proposition 5.19. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then Δ is an identity derivation of X.

Proof: Since Δ is regular $(r, l) - \Delta$ of X then (i) $(e * x) * \Delta (y) = (e * y) * \Delta(x), \forall x, y \in X$ and (ii) $\Delta (e) = e$. Now let $y = e$ in (i),

$\Rightarrow (e * x) * \Delta (e) = (e * e) * \Delta (x)$

$\Rightarrow (e * x) * e = e * \Delta (x)$

$\Rightarrow e * x = e * \Delta (x)$

$\Rightarrow e * (e * x) = e * (e * \Delta (x))$

$\Rightarrow x = \Delta (x), \forall x \in X.$

Hence Δ is an identity derivation of X.

Proposition 5.20. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular $(r, l) - \Delta$ of X. Then for all $x, y \in X$, the following are true.

(i) $e * \Delta (x) = e * \Delta (y) \Leftrightarrow \Delta (x) = \Delta (y)$

(ii) $x * \Delta (y) = \Delta (e) = y * \Delta (x) \Leftrightarrow x = y$

(iii) $e * \Delta (x) = y \Leftrightarrow x = e * \Delta (y)$

(iv) $y * (y * \Delta (x)) = \Delta (x)$.

Proof: (i) Let $e * \Delta (x) = e * \Delta (y) \Leftrightarrow e * (e * \Delta (x)) = e * (e * \Delta (y)) \Leftrightarrow \Delta (x) = \Delta (y)$

(ii) Let $x * \Delta (y) = \Delta (e) \Rightarrow x * \Delta (y) = \Delta (x * x) \Rightarrow x * \Delta (y) = x * \Delta (x) \Rightarrow \Delta (y) = \Delta (x) \Rightarrow y = x$.

Again let $y * \Delta (x) = \Delta (e) \Rightarrow y * \Delta (x) = \Delta (y * y) \Rightarrow y * \Delta (x) = y * \Delta (y) \Rightarrow \Delta (x) = \Delta (y) \Rightarrow x = y$

Also if $x = y, x * \Delta (y) = x * \Delta (x) = \Delta (x * x) = \Delta (e)$ and $y * \Delta (x) = y * \Delta (y) = \Delta (y * y) = \Delta (e)$.

$\therefore x * \Delta (y) = \Delta (e) = y * \Delta (x) \Leftrightarrow x = y$.

(iii) Let $e * \Delta (x) = \Delta (y) \Leftrightarrow e * (e * \Delta (x)) = e * \Delta (y) \Leftrightarrow \Delta (x) = e * \Delta (y)$

(iv) Since $(X,*, e), e \in X$, is an $(BF_1)^e$ then $y * (y * x) = x, \forall x, y \in X$. $\Rightarrow \Delta (y * (y * x)) = \Delta (x) \Rightarrow y * \Delta (y * x) = \Delta (x) \Rightarrow y * (y * \Delta (x)) = \Delta (x)$ $\therefore y * (y * \Delta (x)) = \Delta (x), \forall x, y \in X$.

Theorem 5.21. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ is identity $(r, l) - \Delta$ of X. Then for all $x, y \in X$, the following holds good.

(I) $\Delta (x * x) = e$, *(II)* $\Delta (x * e) = x$, *(BF)* $\Delta (e * (y * x)) = x * y$, *(BG)* $\Delta ((x * y) * (e * y)) = x$, *(BH)* $\Delta (x * y) = e = \Delta (y * x) \Rightarrow x = y$, *(E)* $\Delta (x * (e * y)) = \Delta (y * (e * x))$.

Proof: Straightforward.

Theorem 5.22. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X such that $\Delta (x) * \Delta (z) = e$, and $(z * y) * (x * y) = z * x$, for all $x, y, z \in X$. Then for all $x, y, z \in X$, $((\Delta (x) * \Delta (z)) * (\Delta (y) * \Delta (z))) * (\Delta (x) * \Delta (y)) = e$.

Proof: Consider, $((\Delta (x) * \Delta (z)) * (\Delta (y) * \Delta (z))) * (\Delta (x) * \Delta (y))$

$= (e * (\Delta (y) * \Delta (z))) * (\Delta (x) * \Delta (y)) = (\Delta (z) * \Delta (y)) * (\Delta (x) * \Delta (y))$

$= (\Delta o \Delta o \Delta o \Delta) ((z * y) * (x * y)) = (\Delta o \Delta o \Delta o \Delta) (z * x) = (\Delta o \Delta o \Delta o \Delta) ((z * e) * (x * e))$

$= (\Delta (z) * \Delta (e)) * (\Delta (x) * \Delta (e)) = (\Delta (z) * e) * (\Delta (x) * e) = \Delta (z) * \Delta (x)$

$= e * (\Delta (x) * \Delta (z)) = e * e = e$

$\therefore ((\Delta (x) * \Delta (z)) * (\Delta (y) * \Delta (z))) * (\Delta (x) * \Delta (y)) = e, \forall x, y, z \in X$.

Theorem 5.23. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X such that $\Delta(x) * \Delta(y) = e$ and $(x * z) * (y * z)= x * y$, for all $x, y \in X$. Then $((\Delta (x) * \Delta (z)) * (\Delta (y) * \Delta (z))) * (\Delta (x) * \Delta (y)) = e$, for all $x, y, z \in X$.

Proof: Proof is similar to the proof of theorem 5.22.

Theorem 5.24. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X such that $(x * z) * (y * z) = x * y$, for all $x, y, z \in X$. Then $(\Delta (x) * \Delta (z)) * (\Delta (y) * \Delta (z)) = \Delta (x) * \Delta (y)$, for all $x, y, z \in X$.

Proof: Proof is similar to the proof of theorem. 5.22.

Theorem 5.25. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X with $\Delta (z) = \Delta (x) * \Delta (y)$, for all $x, y, z \in X$. Then $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y)) = \Delta (y) * \Delta (x)$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y))= (\Delta o \Delta o \Delta o \Delta) ((z * x) * (e * (y * z)))$
$= (\Delta o \Delta o \Delta o \Delta) ((y * z) * (e * (z * x)))= ((\Delta (y) * \Delta (z)) * (e * (\Delta (z) * \Delta (x)))$
$= ((\Delta (y) * \Delta (z)) * (e * (e * \Delta (y)))= ((\Delta (y) * \Delta (z)) * \Delta (y) = e * (\Delta (y) * (\Delta (y) * \Delta (z)))$
$= e * (\Delta o \Delta o \Delta) (y * (y * z))= e * (\Delta o \Delta o \Delta) (z)= e * (\Delta o \Delta) (\Delta (z)) = (\Delta o \Delta) (e * \Delta (z))$
$= e * \Delta (z)= \Delta (y) * \Delta (x)$
$\therefore (\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y))= \Delta (y) * \Delta (x), \forall x, y, z \in X.$

Theorem 5.26. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X with $\Delta (z) = \Delta (x) * \Delta (y)$, for all $x, y, z \in X$ and (G). Then $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y)) = \Delta (y) * \Delta (x)$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y))= e * (\Delta (z) * \Delta (y)) * (\Delta (z) * \Delta (x))$
$= (\Delta o \Delta o \Delta o \Delta) (e * ((z * y) * (z * x)))= (\Delta o \Delta o \Delta o \Delta) ((y * z) * (x * z))$
$= (\Delta (y) * \Delta (z)) * (\Delta (x) * \Delta (z))= (\Delta (y) * \Delta (z)) * \Delta (y)= e * (\Delta (y) * (\Delta (y) * \Delta (z)))$ $= e * (\Delta o \Delta o \Delta) (y * (y * z))= e * (\Delta o \Delta) (\Delta (z))= (\Delta o \Delta) (e * \Delta (z)) = e * \Delta (z)= \Delta (y) * \Delta (x).$
$\therefore (\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y))= \Delta (y) * \Delta (x), \forall x, y, z \in X.$

Theorem 5.27. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X with $\Delta (z) = \Delta (x) * \Delta (y)$, for all $x, y, z \in X$. Then $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y)) = \Delta (y) * \Delta (x)$, for all $x, y, z \in X$.

Proof: Consider, $(\Delta (z) * \Delta (x)) * (\Delta (z) * \Delta (y))= (\Delta (x) * \Delta (y)) * \Delta (x)) * (\Delta (z) * \Delta (y))$
$= (e * (\Delta (x) * (\Delta (x) * \Delta (y))) * (\Delta (z) * \Delta (y))= (e * \Delta (y)) * (\Delta (z) * \Delta (y))$
$= e * ((\Delta (z) * \Delta (y)) * (e * \Delta (y)))= e * \Delta (z)= e * (\Delta (x) * \Delta (y) = \Delta (y) * \Delta (x)$
Hence, $(\Delta (z)*\Delta (x)) * (\Delta (z) * \Delta (y))= \Delta (y) * \Delta (x), \forall x, y, z \in X.$

Theorem 5.28. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X. Then $\Delta (y)*(\Delta (y) * \Delta (x)) = \Delta (x)$, for all $x, y \in X$.

Proof: Consider, $\Delta (y) * (\Delta (y) * \Delta (x)) = (\Delta o \Delta o \Delta) (y * (y * x)) = (\Delta o \Delta o \Delta) (e * ((y * x) * y))$
$= (\Delta o \Delta o \Delta) (e * ((y * x)) * (e * y)) = (\Delta o \Delta) (\Delta (x * e)) = \Delta (x * e)) = \Delta (x)$
$\therefore \Delta (y) * (\Delta (y) * \Delta (x)) = \Delta (x), \forall x, y \in X.$

Theorem 5.29. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X with $(z * x) * (z * y) = y* x$. Then $(\Delta (z) * \Delta (x))*(\Delta (z) * \Delta (y)) = \Delta (y)* \Delta (x)$, for all $x, y, z \in X$.

Proof: Proof is similar to the proof of theorem 5.22.

Theorem 5.30. Let $(X,*, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X Then $\Delta (y) * (\Delta (y) * \Delta (x)) = \Delta (x)$, for all $x, y \in X$.

Proof: Consider, $\Delta(y) * (\Delta(y) * \Delta(x)) = (\Delta \circ \Delta \circ \Delta)(y * (y * x)) = (\Delta \circ \Delta \circ \Delta)(e * ((y * x) * y)) = (\Delta \circ \Delta \circ \Delta)(e * ((y * x)) * (e * y)) = (\Delta \circ \Delta)(\Delta(x * e)) = \Delta(x * e)) = \Delta(x)$. $\therefore \Delta(y) * (\Delta(y) * \Delta(x)) = \Delta(x), \forall x, y \in X$.

Theorem 5.31. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X such that $(x * z) * (y * z) = x * y$, for all $x, y, z \in X$. Then $(\Delta(x) * \Delta(z)) * (\Delta(y) * \Delta(z)) = \Delta(x) * \Delta(y)$, for all $x, y, z \in X$.

Proof: Proof is similar to the proof of theorem 5.22.

Theorem 5.32. Let $(X, *, e)$ be an $(BF_1)^e$ and Δ be the regular derivation of X such that $(z * x) * (z * y) = y * x$ and $(\Delta(z) * \Delta(x)) * (\Delta(z) * \Delta(y)) = \Delta(y) * \Delta(x)$, for all $x, y, z \in X$.

Proof: Similar proof as done in theorem 5.22.

6. CONCLUSION

In the present work, the concepts of *(l, r)* and *(r, l) - derivations* were introduced on *e-commutative* BF_1*-algebra* and further extended to the concepts of *regular* and *identity derivations.* In future work authors introduce the concepts of *fuzzy derivations, fuzzy intuitionistic derivations* and *fuzzy cubic derivations.*

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