Journal of Computer and Mathematical Sciences, *Vol.10(3),502-509 March 2019* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Fixed Point Theorems Using A-compatible and A-weak Reciprocally Continuous Mappings in d-Metric Space

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(Received on: March 12, 2019)

ABSTRACT

The purpose of this paper is to prove a common fixed point theorem using A-Compatible and A-weak reciprocally continuous mappings in dislocated metric space. This is a weaker form of the result of Jha and Panthi. We substantiate our theorem with a valid example.

AMS (2010) Mathematics Classification: 54H25, 47H10.

Keywords: Fixed point, A-Compatible, S-Compatible, A-weak reciprocally continuous and S-weak reciprocally continuous mappings, weakly compatible.

1. INTRODUCTION

A first ever fixed point theorem is proved under contraction mapping in a metric space by S. Banach in 1922. Thereafter several fixed point theorems came into light by different authors. The notion of dislocated metric space in which self distance of a point need not be equal to zero was introduced by P. Hitzler and A.K. Seda[1] in 2000. They also generalized the famous Banach contraction principle in this space. Dislocated metric space satisfying certain contractive conditions has been centre of research activity.

In the recent past the concept of reciprocally continuous mappings was introduced by R.P. Pant. Recently Pant, R.K. Bisht and D. Arora[6] improved the notion of reciprocal continuity by introducing weak reciprocal continuity and observed that weak reciprocal continuity is applicable to compatible as well as noncompatible mappings.

In this paper we prove common fixed point theorems for four self maps in which one pair is A-compatible (or) S-compatible and A-weak reciprocally continuous (or) S-weak reciprocally continuous mappings and other pair is weakly compatible.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions

(d_1) $d(x,y) = d(y,x)$

(d_2) $d(x,y) = d(y,x) = 0$ implies $x = y$.

(d_3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Then d is called *dislocated metric* on X and the pair (X,d) is called a *dislocated metric space*(or shortly d-metric space)

Definition2.2. A sequence $< x_n >$ in a d-metric space (X,d) is called a *Cauchy sequence* if for given $\varepsilon > 0$, there exists $n_0 \in >$ such that for all $m,n \ge n_0$, we have $d(x_m, x_n) < \varepsilon$.

Definition2.3. A sequence $< x_n >$ in a d-metric space (X,d) *converges* if there exists $x \in X$ such that $d(x_n, x) \to 0$.

Definition2.4. A d-metric space (X,d) is called *complete* if for every Cauchy sequence is convergent.

Definition2.5. A and S are two self maps of a d-metric space (X,d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$,whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition2.6. A and S are two self maps of a d-metric space (X,d) are said to be *weakly compatible mappings* if they commute at their coincidence point. That is if $Au = Su$ for some $u \in X$ then $ASu = SAu$.

Definition2.7. A and S are two self maps of a d-metric space (X,d) are said to be weak reciprocally continuous mappings if $\lim_{n\to\infty} ASx_n = Az$ or $\lim_{n\to\infty} SAx_n = Sz$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition2.8. A and S are two self maps of a d-metric space (X,d) are said to be A-weak reciprocally continuous mappings if $\lim_{n\to\infty} ASx_n = Az$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition2.9. A and S are two self maps of a d-metric space (X,d) are said to be S-weak reciprocally continuous mappings if $\lim_{n\to\infty} SAx_n = Sz$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition2.10. A and S are two self maps of a d-metric space (X,d) are said to be *A-compatible* if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$,whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition2.11. A and S are two self maps of a d-metric space (X,d) are said to be *S-compatible* if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$,whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

In this section, we prove fixed point theorems for A-compatible and A-weak reciprocally continuous mappings in dislocated metric spaces. Our result improves, extends and generalizes the result of[5].

3. MAIN RESULT

3.1 Theorem: Let A, B, S and T be self mappings from a complete d-metric space (X,d) into itself satisfying the following conditions

$T(X) \subset A(X)$ and $S(X) \subset B(X)$ (3.1.1)

$d(Sx,Ty) \le \alpha d(Ax,Ty) + \beta d(By,Sx) + \gamma d(Ax,By)$ (3.1.2)

for all $x, y \in X$ where $\alpha, \beta, \gamma \ge 0, 0 \le \alpha + \beta + \gamma < \frac{1}{2}$.

the pair (S,A) is A-weak reciprocally continuous and A-compatible (3.1.3)

and the pair (T,B) is weakly compatible . (3.1.4)

Then A,B,S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point in X such that $Tx_0 \in T(X) \subset A(X)$ gives $Tx_0 = Ax_1$ for some $x_1 \in X$, $Sx_1 \in S(X) \subset B(X)$, there exists a point $x_2 \in X$ such that $Sx_1 = Bx_2$. Again $Tx_2 \in T(X) \subset A(X)$ gives $Tx_2 = Ax_3$ for some $x_3 \in X$. Now $Sx_3 \in S(X) \subset B(X)$ gives $Sx_3 = Bx_4$ and so on. Proceeding in this fashion, choose $x_n \in X$ such that

$Tx_{2n} = Ax_{2n+1}$ and $Sx_{2n+1} = Bx_{2n+2}$, for $n \geq 0$. We consider the sequence $\{y_n\}$ defined by $y_{2n} = Tx_{2n} = Ax_{2n+1}$ and $y_{2n+1} = Sx_{2n+1} = Bx_{2n+2}$ for $n = 0,1,2,......$ (3.1.5)

We claim that $\{y_n\}$ is a Cauchy sequence.

Using the conditions (3.1.1), (3.1.2) and (3.1.5), we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$
$$\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Sx_{2n}) + \gamma d(Ax_{2n}, Bx_{2n+1})$$
$$\leq \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n}, y_{2n-1}) + \gamma d(y_{2n-1}, y_{2n})$$
$$\leq \alpha[d(y_{2n-1}, y_{2n}) + (y_{2n}, y_{2n+1})] + \beta d(y_{2n}, y_{2n-1}) + \gamma d(y_{2n-1}, y_{2n})$$
$$= (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + \alpha(y_{2n}, y_{2n+1})$$

this implies

$$(1-\alpha) d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n})$$

which gives

$$d(y_{2n}, y_{2n+1}) \leq \frac{(\alpha + \beta + \gamma)}{(1-\alpha)} d(y_{2n-1}, y_{2n})$$

this implies

$$d(y_{2n}, y_{2n+1}) \leq \lambda\, d(y_{2n-1}, y_{2n}) \text{ where } \lambda = \frac{\alpha + \beta + \gamma}{1-\alpha} < 1.$$

This shows that,

$$d(y_n, y_{n+1}) \leq \lambda\, d(y_{n-1}, y_n) \leq .... \leq \lambda^n d(y_0, y_1).$$

Now for every integer $m > 0$, we get

$$d(y_n, y_{n+m}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+m-1}, y_{n+m})$$
$$\leq \lambda^n d(y_0, y_1) + \lambda^{n+1} d(y_0, y_1) + ... + \lambda^{n+m-1} d(y_0, y_1)$$
$$\leq \lambda^n + \lambda^{n+1} + ... + \lambda^{n+m-1} \; d(y_0, y_1)$$
$$\leq \lambda^n \; 1 + \lambda + \lambda^2 + ... + \lambda^{m-1} \; d(y_0, y_1)$$

Since $0 < \lambda < 1$, $\lambda^n \to 0$ as $n \to \infty$, so that $d(y_n, y_{n+m}) \to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete dislocated metric space, it converges to some point, say $z \in X$.

Therefore, the subsequences,

$$\{Sx_{2n}\} \to z, \{Bx_{2n+1}\} \to z, \{Tx_{2n+1}\} \to z \text{ and } \{Ax_{2n+2}\} \to z. \quad (3.1.6)$$

Since the pair (S,A) is A-weakly reciprocally continuous implies $\lim_{n\to\infty} ASx_{2n} = Az$. (3.1.7)

Also the pair (S,A) is A-compatible mapping then $\lim_{n\to\infty} d(ASx_{2n}, SSx_{2n}) = 0$.

This gives $\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} SSx_{2n}$. (3.1.8)

From the conditions (3.1.7) and (3.1.8), we have

$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} SSx_{2n} = Az.$ (3.1.9)

Put $x = Sx_{2n}, y = x_{2n+1}$ in the condition (3.1.2), we get

$d(SSx_{2n}, Tx_{2n+1}) \le \alpha d(ASx_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, SSx_{2n}) + \gamma d(ASx_{2n}, Bx_{2n+1})$

letting $n \to \infty$ and using the conditions (3.1.6) and (3.1.9), we have

$d(Az, z) \le \alpha d(Az, z) + \beta d(z, Az) + \gamma d(Az, z)$

this implies

$d(Az, z) \le (\alpha + \beta + \gamma) d(Az, z)$

this gives

$(1 - \alpha - \beta - \gamma) d(Az, z) \le 0$, which is a contradiction since $\alpha + \beta + \gamma < \frac{1}{2}$ and this implies $Az = z$.

Since $S(X) \subset B(X)$ implies there exists $u \in X$ such that $z = Bu$. (3.1.10)

Put $x = x_{2n}, y = u$ in (3.1.2), we get

$d(Sx_{2n}, Tu) \le \alpha d(Ax_{2n}, Tu) + \beta d(Bu, Sx_{2n}) + \gamma d(Ax_{2n}, Bu)$

letting $n \to \infty$ and using the conditions (3.1.6) ,(3.1.10) and $Az = z$, we have

$d(z, Tu) \le \alpha d(z, Tu) + \beta d(z, z) + \gamma d(z, z)$

this implies

$d(z, Tu) \le \alpha d(z, Tu) + \beta[d(z, Tu) + d(Tu, z)] + \gamma[d(z, Tu) + d(Tu, z)]$

this gives

$d(z, Tu) \le (\alpha + 2\beta + 2\gamma) d(z, Tu)$

this implies

$(1 - \alpha - 2\beta - 2\gamma) d(z, Tu) \le 0$, which is a contradiction since $\alpha + \beta + \gamma < \frac{1}{2}$ and this gives $z = Tu$.

Hence, we have $Bu = Tu = z$. (3.1.11)

Since the pair (T,B) is weakly compatible then $BTu = TBu$
and this implies $Bz = Tz$. (3.1.12)

Put $x = x_{2n}, y = z$ in (3.1.2), we get

$d(Sx_{2n}, Tz) \le \alpha d(Ax_{2n}, Tz) + \beta d(Bz, Sx_{2n}) + \gamma d(Ax_{2n}, Bz)$

letting $n \to \infty$ and using the conditions (3.1.6) and (3.1.12), we have

$d(z,Tz) \le \alpha d(z,Tz) + \beta d(Tz,z) + \gamma d(z,Tz)$

$(1-\alpha-\beta-\gamma)d(z,Tz) \le 0$, which is a contradiction since $\alpha+\beta+\gamma < \frac{1}{2}$. So, we have $Tz = z$. Hence $Az = Tz = Bz = z$. (3.1.13)

Put $x = z, y = x_{2n}$ in (3.1.2), we get

$d(Sz,Tx_{2n}) \le \alpha d(Az,Tx_{2n}) + \beta d(Bx_{2n},Sz) + \gamma d(Az,Bx_{2n})$

letting $n \to \infty$ and using the conditions (3.1.6) and (3.1.13), we have

$d(Sz,z) \le \alpha d(z,z) + \beta d(z,Sz) + \gamma d(z,z)$

$d(Sz,z) \le \beta d(z,Sz) + \alpha[d(z,Sz) + d(Sz,z)] + \gamma[d(z,Sz) + d(Sz,z)]$

$(1-2\alpha-\beta-2\gamma)d(z,Sz) \le 0$, which is a contradiction since $\alpha+\beta+\gamma < \frac{1}{2}$. So, we have $Sz = z$.

Therefore $Az = Bz = Sz = Tz = z$.

Since $Az = Sz = Bz = Tz = z$, we get z is a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

Theorem 3.2. Let A,B,S and T be self mappings of a complete metric space (X,d) satisfying

(3.2.1) $T(X) \subset A(X)$ and $S(X) \subset T(X)$

(3.2.2) $d(Sx,Ty) \le \alpha d(Ax,Ty) + \beta d(By,Sx) + \gamma d(Ax,By)$

for all $x, y \in X$ where $\alpha,\beta,\gamma \ge 0, 0 \le \alpha+\beta+\gamma < \frac{1}{2}$

(3.2.3) the pair (S,A) is S-weak reciprocally continuous and S-compatible and

(3.2.4) the pair (T,B) is weakly compatible .

Then A,B,S and T have a unique common fixed point in X.

Proof: The proof of this theorem is similar to the Theorem3.1.

Our main Theorem3.1 can be validating by using the following example.

3.3 Example: Let $X = [0,1]$ with dislocated metric $d(x,y) = |x-y|$. Define self maps of A, B, S and T of X by

$$A(x) = B(x) = \begin{cases} \frac{1}{6} & if\ 0 \le x < \frac{1}{2} \\ 1-x & if\ \frac{1}{2} \le x \le 1 \end{cases} \quad \text{and} \quad S(x) = T(x) = \begin{cases} \frac{1}{4} & if\ 0 \le x < \frac{1}{2} \\ \frac{1}{2} & if\ \frac{1}{2} \le x \le 1 \end{cases}$$

Now $A(X) = B(X) = \left[0, \frac{1}{2}\right]$ while $S(X) = T(X) = \left\{\frac{1}{4}, \frac{1}{2}\right\}$ so that the conditions $T(X) \subset A(X)$ and $S(X) \subset B(X)$ are satisfied.

Now we prove that the pair (S,A) is A-compatible and A-weak reciprocally continuous and the pair (T,B) is weakly compatible. For this, define a sequence $\{x_n\}$ where $x_n = \frac{1}{2}+\frac{1}{3n}$ for $n \geq 1$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} \text{ and } \lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left[S\left(\frac{1}{2}+\frac{1}{3n}\right)\right] = \lim_{n\to\infty} A\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Also $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} S\left[S\left(\frac{1}{2}+\frac{1}{3n}\right)\right] = \lim_{n\to\infty} S\left(\frac{1}{2}\right) = \frac{1}{2}$.

Therefore $\lim_{n\to\infty} d(ASx_n, SSx_n) = d\left(\frac{1}{2},\frac{1}{2}\right) = 0$ implies that the pair (S,A) is A-compatible.

Further $\lim_{n\to\infty} ASx_n = \frac{1}{2} = A\left(\frac{1}{2}\right)$ which implies that the pair (A,S) is A-weak reciprocally continuous.

Since $B\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$ and $TB\left[\frac{1}{2}\right] = BT\left[\frac{1}{2}\right] = \frac{1}{2}$

Therefore $TB\left(\frac{1}{2}\right) = BT\left(\frac{1}{2}\right) = \frac{1}{2}$, showing that B and T are weakly compatible

3.3 CONCLUSION

From the example given above, we observe that the pair (S, A) is A- weak reciprocally continuous and A-compatible and other pair (T, B) is weakly compatible which are weaker conditions than the compatibility of the pairs (S,A) and (B,T). Further the continuity of any one of the mapping is being dropped in Theorem3.1. Also the condition (3.1.2) holds for the values of $\alpha, \beta, \gamma \geq 0$ where $0 \leq \alpha+\beta+\gamma < \frac{1}{2}$. Also from the example we observe $A\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$, showing that $\frac{1}{2}$ is a common fixed point of A, B, S and T. In fact '$\frac{1}{2}$' is the unique common fixed point of A, B, S and T.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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