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Equitable Total Domination in Graphs

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ABSTRACT

A subset D of a vertex set $V(G)$ of a graph $G = (V, E)$ is called an equitable dominating set if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ and $deg(v)$ are denoted as the degree of a vertex u and v respectively. The equitable domination number of a graph $\gamma^e(G)$ of G is the minimum cardinality of an equitable dominating set of G. An equitable dominating set D is said to be an equitable total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices. The equitable total domination number $\gamma_t^e(G)$ of G is the minimum cardinality of an equitable total dominating set of G. In this paper, we initiate a study on new domination parameter equitable total domination number of a graph, characterization is given for equitable total dominating set is minimal and also discussed Northaus-Gaddum type results.

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Keywords: total domination number, equitable domination number, equitable total domination number.

1. INTRODUCTION

The *graph* $G = (V, E)$ we mean a finite, undirected with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary[3] and Haynes *et al.*[4]

For any vertex $v \in V$, the open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. Degree of a vertex v is denoted by $deg(v)$. The *maximum* (*minimum*) degree of G is denoted by $\Delta(G)$ ($\delta(G)$).

The *diameter* of a connected graph is the maximum distance between two vertices in G, and is denoted by $diam(G)$. The length of a shortest cycle is the *girth* of G and is denoted by $g(G)$. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Northaus-Gaddum type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement.

A subset D of V is called a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a minimal dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. Various types of domination parameters have been defined and studied by several authors and more than seventy five models of domination parameters are listed in the appendix of Haynes *et al.*[4] Cockayne *et al.*[2] introduced the concept of total domination in graphs. A dominating set D of G is called a *total dominating set* if $\langle D \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$.

Swaminathan *et al.*[10] introduced the concept of equitable domination in graphs, by considering the following real world problems; In a network nodes with nearly equal capacity may interact with each other in a better way. In this society persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Equitability among citizens in terms of wealth, health, status etc is the goal of a democratic nation.

In order to study this practical concept a graph model is to be created and defined as follows:

A subset D of V is called an *equitable dominating set* if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set of G is called the *equitable* domination number of G and is denoted by $\gamma^e(G)$.

In this paper, we use this idea to develop the concept of equitable total dominating set and equitable total domination number of a graph.

A subset D of V is called an equitable total dominating set of G, if D is an equitable dominating set and $\langle D \rangle$ has no isolated vertices. The minimum cardinality taken over all equitable total dominating sets is the equitable total domination number and is denoted by $\gamma_t^e(G)$[8].

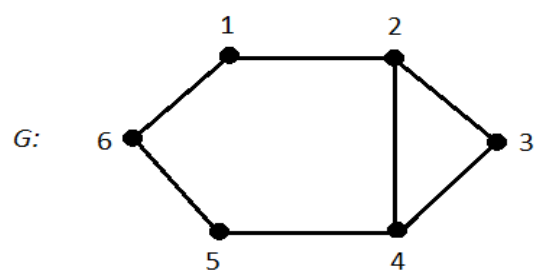

Figure 1

The equitable total dominating sets of graph G in Figure 1, are: $D_1 = \{2, 3, 5, 6\}$, $D_2 = \{1, 3, 4, 6\}$, $D_3 = \{1, 2, 4\}$ and $D_4 = \{2, 4, 5\}$. Therefore, $\gamma_t^e(G) = 3$.

In this paper, we initiate a study on equitable total dominating sets in graphs. This parameter is analogous to the total domination number which has already been studied by Cockayne, Dawes and Hedetniemi in[2].

2. MAIN RESULTS

By the definition of an equitable total dominating set, the following result is obvious.

Theorem 2.1

A total dominating set D of G is an equitable total dominating set if and only if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$.

Next, we list the exact values of $\gamma_t^e(G)$ for some standard class of graphs.

Theorem 2.2

1. For any path P_p with $p \geq 2$ vertices, $$\gamma_t^e(P_p) = \begin{cases} \frac{p}{2} & ; \text{ if } p \equiv 0 \pmod 4 \\ \left\lfloor \frac{p}{2} \right\rfloor + 1 & ; \text{ otherwise}. \end{cases}$$
2. For any cycle C_p with $p \geq 3$ vertices, $$\gamma_t^e(C_p) = \begin{cases} \left\lceil \frac{p}{2} \right\rceil & ; \text{ if } p \not\equiv 2 \pmod 4 \\ \frac{p+2}{2} & ; \ p \equiv 0 \pmod 4. \end{cases}$$
3. For any complete graph K_p with $p \geq 2$ vertices, $\gamma_t^e(K_p) = 2$.
4. For any wheel W_p with $p \geq 4$ vertices, $$\gamma_t^e(W_p) = \begin{cases} 2 & ; \text{ if } p = 4, 5 \\ \left\lceil \frac{p-1}{3} \right\rceil + 1 & ; \text{ otherwise}. \end{cases}$$
5. For any complete bipartite graph $K_{m,n}$, $$\gamma_t^e(K_{m,n}) = \begin{cases} 2 & ; \text{ if } |m-n| \leq 1, 1 \leq m \leq n \\ m+n & ; \text{ if } |m-n| \geq 2, m, n \geq 2. \end{cases}$$
6. For any star $K_{1,p-1}$ with $p \geq 2$ vertices, $\gamma_t^e(K_{1,p-1}) = p$.

Proof

(i) As the degree of any vertex of P_p is either 1 or 2. Clearly, any total dominating set is equitable. Therefore $\gamma_t^e(P_p) = \gamma_t(P_p)$.

(ii) As C_p is a connected 2 - regular graph, any total dominating set is equitable. Hence $\gamma_t(C_p) = \gamma_t^e(C_p)$.

(iii) For any complete graph K_p with $p \geq 2$, any two vertices of K_p forms an equitable total dominating set of K_p. Therefore $\gamma_t^e(K_p) = 2$.

(iv) Let W_p be a wheel, such that $V(W_p) = \{u, v_1, v_2, \cdots, v_{p-1}\}$, where u is the center vertex and each v_i; $i = 1, \cdots, p-1$ is on the cycle. Therefore the $deg_{W_p}(v_i) = 3$, where $1 \leq i \leq p-1$ and $deg_{W_p}(u) = p-1$, hence $p \geq 4$.

We consider the following cases:

Case1. Let p = 4 or 5.

Then $deg_{W_p}(u) = p - 1 \leq 4$ and $deg_{W_p}(v_i) = 3$, for all i, $1 \leq i \leq p-1$.

Therefore $D = \{u\} \cup \{v_i\}$, $1 \leq i \leq p-1$, is an equitable total dominating set of W_p.

Hence $\gamma_t^e(W_p) = |D|$
$= |\{u\} \cup \{v_i\}| = 2.$

Case2. Let $p \geq 6$.

In this case $deg_{W_p}(u) \geq 5$, while $deg_{W_p}(v_i) = 3$ for all i, where $1 \leq i \leq p-1$.

However $D = \{u\} \cup \{v_i\}$ is a total dominating set, but not equitable total dominating set.

If $$D = \begin{cases} \{u, v_1, v_4, \cdots v_{3k-2}\}, & if \ p-1 = 3k \\ \{u, v_1, v_4, \cdots v_{3k-2}, v_{3k-1}\}, & if \ p-1 = 3k+1 \\ \{u, v_1, v_4, \cdots v_{3k-2}, v_{3k+1}\}, & if \ p-1 = 3k+2 \end{cases}$$

then for any $v_i \in V - D$, there exists v_{i-1} or $v_{i+1} \in D$ such that, $v_i v_{i-1}$ or $v_i v_{i+1} \in E(G)$ and $deg(v_i) = deg(v_{i-1}) = 3$ or $deg(v_i) = deg(v_{i+1}) = 3$. Therefore D is an equitable total dominating set of W_p.

Now,

$$|D| = \begin{cases} k+1, & if \;\; p = 3k \\ k+2, & if \;\; p = 3k+1 \;\; or \;\; 3k+2 \,. \end{cases}$$

Also, when $p - 1 = 3k$, $\lceil \frac{p-1}{3} \rceil = k$,

when

$p - 1 = 3k + 1$ or $3k + 2$, $\lceil \frac{p-1}{3} \rceil = k + 1$.

Hence, $|D| = \lceil \frac{p-1}{3} \rceil + 1$.

Therefore $\gamma_t^e(W_p) = \lceil \frac{p-1}{3} \rceil + 1$.

(v) Let $K_{m,n}$ be the complete bipartite graph with m vertices in one partition say V_1 and n vertices in another partion say V_2.

Then, $deg_{K_{m,n}}(u) = \begin{cases} m, & if \;\; u = v_1 \\ n, & if \;\; u = v_2 \end{cases}$.

If $|m - n| \leq 1$, then for any vertex $u \in V_1$ and $v \in V_2$ constitute a dominating set which is an equitable total dominating set.

Therefore, $\gamma_t^e(K_{m,n}) = 2$ for $|m - n| \leq 1$.

If $|m - n| \geq 2$ and $|D| < m + n$, where D is a minimum equitable total dominating set of $K_{m,n}$, then there exists $u \in V - D$. Let $u \in V_1$.

Therefore $deg_{K_{m,n}}(u) = n$.

There is a vertex $v \in D$, such that v is adjacent to $u \in V_1$ and $|deg_{K_{m,n}}(v) - deg_{K_{m,n}}(u)| \leq 1$. Since V_1 is independent, v must belongs to V_2. Therefore $deg_{K_{m,n}}(v) = m$. Hence $|deg_{K_{m,n}}(v) - deg_{K_{m,n}}(u)| = |m - n| \geq 2$, which is a contradiction.

Therefore $|D| = m + n$.

Hence $\gamma_t^e(K_{m,n}) = m + n$ for $|m - n| \geq 2$ for all $n, m \geq 2$.

(vi) If $G = K_{1,p-1}; p \geq 2$, then clearly $\gamma_t(G) = 2$.

By the definition of equitable total dominating set, D should contain all the vertices of G. Therefore, $\gamma_t^e(K_{1,p-1}) = p$; $p \geq 2$.

From the above results the following bound is immediate.

Theorem 2.3. *For any graph* G *with no isolates* , $2 \leq \gamma_t^e(G) \leq p$.

Corollary 2.1. *For any graph* $G(\neq K_{m,n}; |m - n| \geq 2 \,;\, m, n \geq 2)$ *without isolated vertices*, $\gamma_t^e(G) \leq \frac{p}{2}$.

Definition 1. An equitable total dominating set is said to be *minimal equitable total dominating set* if no proper subset of D is an equitable total dominating set.

Next theorem gives the characterization of the minimal equitable total dominating set.

Theorem 2.4. For any graph G without isolated vertices, an equitable total dominating set D is minimal if and only if for every u ∈ D, one of the following two properties holds:

(i) There exists a vertex $v \in V - D$ *such that* $N(v) \cap D = \{u\}$, $|deg(u) - deg(v)| \leq 1$.

(ii) $\langle D - \{u\} \rangle$ *contains no isolated vertices.*

Proof. Assume that D is a minimal equitable total dominating set and (i) and (ii) do not hold. Then for some $u \in D$, there exists $v \in V - D$ such that $|deg(u) - deg(v)| \leq 1$ and for every $v \in V - D$, either $N(v) \cap D \neq \{u\}$ or $|deg(u) -$

$deg(v)| \geq 2$ or both. Therefore $\langle D-\{u\}\rangle$ contains an isolated vertex, contradiction to the minimality of D. Therefore (i) and (ii) holds.

Conversely, if for every vertex $u \in D$, the statement (i) or (ii) holds and D is not minimal. Then there exists $u \in D$ such that $D-\{u\}$ is an equitable total dominating set. Therefore there exists $v \in D-\{u\}$ such that v equitably dominates u. That is, $v \in N(u)$ and $|deg(u)-deg(v)| \leq 1$. Hence u does not satisfy (i). Then u must satisfy (ii) and there exists $v \in V-D$ such that $N(v) \cap D=\{u\}$ and $|deg(u)-deg(v)| \leq 1$. And also there exists $w \in D-\{u\}$ such that w is adjacent to v. Therefore $w \in N(v) \cap D$, $|deg(w)-deg(v)| \leq 1$ and $w \neq u$, a contradiction to $N(v) \cap D=\{u\}$. Hence D is a minimal equitable total dominating set.

Proposition 2.1. *For any graph G without isolated vertices, $\gamma_t(G) \leq \gamma_t^e(G)$.*

Proof. Every equitable total dominating set is a total dominating set. Thus $\gamma_t(G) \leq \gamma_t^e(G)$.

Theorem 2.5. *If G is a r-regular for $r \geq 1$ or (k, k+1) bi-regular for any positive integer k, then $\gamma_t^e(G)=\gamma_t(G)$.*

Proof. Suppose G is a regular graph. Then every vertex of G is of same degree say k. Let D be the minimal total dominating set of G, then $|D|=\gamma_t(G)$. If $u \in V-D$, then D is a total dominating set, then there exists $v \in D$ and $uv \in E(G)$, also $deg(u)=deg(v)=k$. Therefore $|deg(u)-deg(v)|=0 \leq 1$. Hence D is an equitable total dominating set of G, such that $\gamma_t^e(G) \leq |D|=\gamma_t(G)$. And also we have $\gamma_t(G) \leq \gamma_t^e(G)$. Therefore $\gamma_t(G)=\gamma_t^e(G)$.

Now, suppose G is a $(k, k+1)$ bi-regular graph. Then the degree of each vertex in G is either k or $k+1$, where k is a positive integer. Let D be a minimal total dominating set of G, i.e $|D|=\gamma_t(G)$ and $u \in V-D$, then there exists $v \in D$ such that $uv \in E(G)$ and one of the vertex u or v is with degree k and other is with degree $k+1$. This implies $|deg(u)-deg(v)|=1$. Therefore D is an equitable total dominating set of G. Hence $\gamma_t^e(G) \leq |D|=\gamma_t(G)$. But also we have that $\gamma_t(G) \leq \gamma_t^e(G)$. Therefore, $\gamma_t^e(G)=\gamma_t(G)$.

In next theorem we give the upper bound for $\gamma_t^e(G)$ in terms of order and maximum degree of G.

Theorem 2.6. *For any connected graph $G(\neq K_{m,n}; |m-n| \geq 2; m, n \geq 2)$ contains no isolated vertices and $\Delta(G)<p-1$, then $\gamma_t^e(G) \leq p-\Delta(G)$.*

Proof. Let v be a vertex with maximum degree $\Delta(G)$ in G and $X=V-N[v]$. If $X=\phi$, then $\Delta(G)=p-1$ and $\gamma_t^e(G)=p-\Delta(G)+1=2$, but $\Delta(G)<p-1$, therefore $X \neq \phi$. Let $x \in X$ is adjacent to $y \in N(v)$. Let C, Δ' be the vertex set and maximum degree of the component of $G[X]$ which contains x. The component of $G[C]$ has a total equitable dominating set Y of cardinality at most $|C|-\Delta'+1$.

If $\Delta'=1$ then $|C|=2$ and the set $\{v, y, x\} \cup (X-C)$ is equitable total dominating in G and $\gamma_t^e(G) \leq 3+(p-\Delta'(G)-1)-2$

$=p-\Delta(G)$.

If $\Delta'(G)>1$, then the set $\{v, y\} \cup Y \cup (X-C)$ is equitable total dominating in G and $\gamma_t^e(G) \leq 2+(|C|-\Delta'+1)+p-\Delta(G)-1-|C|$.

$= p - \Delta(G) + (2 - \Delta')$
$\leq p - \Delta(G)$.
Therefore $\gamma_t^e(G) \leq p - \Delta(G)$.

Theorem 2.7. *If* $G(\neq K_{m,n}; |m-n| \geq 2; m,n \geq 2)$ *be a graph with* $diam(G) = 2$, *then* $\gamma_t^e(G) \leq \delta(G) + 1$ *and this bound is sharp.*

Proof. Let $v \in V(G)$ and $deg(v) = \delta(G)$. Since $diam(G) = 2$, $N(v)$ is a dominating set in G, $D = N(v) \cup \{v\}$ is an equitable total dominating set of G and $|D| = \delta(G) + 1$. Hence $\gamma_t^e(G) \leq \delta(G) + 1$.

We know that $\gamma_t^e(C_5) = 3$, $\delta(C_5) = 2$, and $diam(C_5) = 2$, then $\gamma_t^e(C_5) = \delta(C_5) + 1$.

In the following theorem we establish the upper bound for $\gamma_t^e(G)$ in terms of order and girth of G.

Theorem 2.8. *For any connected graph* G *with girth* $g(G) \geq 5$, *and* $\delta(G) \geq 2$,
$\gamma_t^e(G) \leq p - \lceil \frac{g(G)}{2} \rceil + 1$.

Proof. Let G be a connected graph with girth $g(G) \geq 5$ and $\delta(G) \geq 2$. Now let us remove the cycle C of shortest length from G to form G'. Suppose an arbitrary vertex $v \in V(G')$, then v has at least two neighbors say x and y. If $x, y \in C$ and $d(x,y) \geq 3$, then by replacing the path from x to y on C with the path xvy which reduces the girth of G, a contradiction. If $d(x,y) \leq 2$, then x, y, v are on either C_3 or C_4 in G, contradiction to the hypothesis that $g(G) \geq 5$. Hence no vertex in G' has two or more neighbors on C.

Therefore, $\gamma_t^e(G) \leq p - \lceil \frac{g(G)}{2} \rceil + 1$.

Theorem A[7]. *For any graph* G *without isolated vertices,* $\gamma(G) \geq \lceil \frac{p}{\Delta(G)+1} \rceil$

Next theorem gives the lower bound for $\gamma_t^e(G)$ in terms of order and maximum degree of G.

Theorem 2.9. *For any nontrivial connected graph* G, $\lceil \frac{p}{1+\Delta(G)} \rceil + 1 \leq \gamma_t^e(G)$.

Next theorem gives Nordhaus-Gaddum type results for $\gamma_t^e(G)$.

Theorem 2.10. *If* $G(\neq K_{m,n}; |m-n| \geq 2; m,n \geq 2)$ *has* p *vertices, no isolated vertices and* $\Delta(G) < p - 1$, *then*
1. $\gamma_t^e(G) + \gamma_t^e(\overline{G}) \leq 2\lceil \frac{p}{2} \rceil$
2. $\gamma_t^e(G) \cdot \gamma_t^e(\overline{G}) \leq (\lceil \frac{p}{2} \rceil)^2$

Further, equality holds for $G = C_4$ and these bounds are sharp.

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