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Common Fixed Point Results for Generalized Expanding Mappings in Cone Metric Space over Banach Algebra

Aziz Ahmed[1] and J. N. Salunke[2]

[1]Matoshri Pratishthan Group of Institutions,
School of Engineering, Nanded, Maharashtra, INDIA.
[2]Swami Ramanandh Teerth Marathwada University, Nanded, INDIA.
email: azizahmed02@gmail.com, drjnsalunke@gmail.com

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ABSTRACT

In this paper, we present coincidence point and unique common fixed point result of generalized expansion mappings without continuity in cone metric space over Banach algebra.

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Keywords: Fixed point, Cone Metric space, Banach algebra, generalized expansion mapping, c-sequence, weakly compatible maps.

1. INTRODUCTION

It seems that the concept of K- metric space was originated in 1934, in the papers of D. Kurepa[13]; he replaced the set of non-negative real numbers with cone K of Banach space E and called it as K-Metric Space.

In 2007, Huang and Zhang[1] extend the same concept introduced by D. Kurepa[13] by replacing the cone K with Banach space E, called such space as Cone Metric Space; also by considering the interior points of the cone they defined convergence.

They proved some fixed point theorems of contractive mappings cone metric space. After that, many authors established fixed point results of contractive mappings in cone metric space. Recently, in 2013 H. Liu and S. Xu[2] announced the notion of cone metric space over Banach algebra by replacing Banach space by Banach algebra and obtained some fixed point theorems of generalized Lipschitz mapping by replacing the constant k with spectral radius $r(k)$ which is weaker condition. In 1984 Wang *et al.*,[10] introduced the concept of expansion mapping on metric space and proved some results.

In this paper we obtained coincidence point and fixed point result for generalized expanding mapping defined by Aziz and Salunke[11] on cone metric space over Banach algebra.

Our results generalized and extend various results present in the literature.

2. PRELIMINARIES

Let A be a real Banach algebra. i.e. A is a real Banach space in which an operation of multiplication is defined subject to the following properties, for all $x, y, z \in A,\ \alpha \in \mathbb{R}$

1. $(xy)z = x(yz)$
2. $x(y + z) = xy + xz$ *and* $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$

In this paper we assume A is a Banach algebra with a unity (i.e. a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An elements $x \in A$ is said to be invertible if there is an inverse element $y \in A$, such that $yx = xy = e$. The inverse of x is denoted by x^{-1}.[3]

Proposition 2.1:([3]) Let A be a Banach algebra with a unity e, and $x \in A$. If the spectral radius $r(x)$ of x is less than 1 i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1.$$

then $e - x$ is invertible . Actually

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now recall the concepts of cone for Banach algebra A. A subset P of Banach algebra A is called a cone of A if

1. P is non-empty closed and $\{e, \theta\} \subset P$;
2. $\alpha P + \beta P \subset P$for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

Where θ denotes the null of the Banach algebra A, for a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P. x \prec y$ will stands for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in A,\ 0 \preccurlyeq x \preccurlyeq y \Rightarrow\ \|x\| \leq K\|y\|$

The least positive number satisfying above inequality is called the normal constant of P [1] .

Definition 2.2 :([2]) Let A be a Banach algebra and $P \subseteq A$ be a cone. Let X be a nonempty set. Suppose that a mapping $d: X \times X \to A$ satisfies the following conditions:

1. $0 \preccurlyeq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$for all $x, y \in$ X
3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in$ X.

Then d is called a cone metric on X and (X, d) is called a cone metric space over Banach algebra A.

Example 2.3 :([2]) Let $A = M_n(\mathbb{R}) = a = \left\{\left(a_{ij}\right)_{n\times n} | \, a_{ij} \in \mathbb{R}, \text{ for all } 1 \le i,j \le n\right\}$ be the algebra of all n-square real matrices, and define the norm

$$\|a\| = \sum_{k1\le i,j\le n} \left|a_{ij}\right|$$

Then A is a real Banach algebra with the unit e the identity matrix.
Let $P = \{a \in A | a_{ij} \ge 0, for\ all\ 1 \le i,j \le n\}$. Then $P \subset A$ is a normal cone with normal constant $K = 1$.
Let $X = M_n(\mathbb{R})$, and define the metric $d: X \times X \to A$ by

$$d(x,y) = d\left(\left(x_{ij}\right)_{n\times n}, \left(y_{ij}\right)_{n\times n}\right) = \left(\left|x_{ij} - y_{ij}\right|\right)_{n\times n} \in A.$$

Then (X,d) is a cone metric space over Banach algebra A with normality.

Example 2.4 : Let $A = C^1_{\mathbb{R}}[0,1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|\acute{x}\|_\infty$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real unital Banach algebra with unity $e = 1$. The set $P = \{x \in A: x \ge 0\}$ is a cone in A. Moreover, P is not normal. Let $X = \{1,2,3\}$. Define $d: X \times X \to A$ by , $d(x,x)(t) = 0$ for $x \in X$
$d(1,2)(t) = d(2,1)(t) = d(1,3)(t) = d(3,1)(t) = e^t$, $\quad d(2,3)(t) = d(3,2)(t) = 0$.
We see that (X,d) is a cone metric space over Banach algebra A .

Definition 2.5: ([1,2])Let (X,d)be a cone metric space over Banach algebra A. Let $\{x_n\}$ be a sequence in X, then:

1. $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$, for all $n \ge N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.
2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$, for all $n, m \ge N$.
3. (X,d) be a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 2.6: ([4]) If E is a real Banach space with a solid cone P and if $\theta \le u \ll c$ for each $\theta \ll c$ then $u = \theta$.

Lemma2.7: ([4]) If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ $(n \to \infty)$, then for any $\theta \ll c$ there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.

Remark 2.8: ([5]) If $r(k) < 1$ then $\|k^n\| \to 0$ $(n \to \infty)$.

Lemma 2.9: ([6]) The limit of a convergent sequence in a cone metric space is unique.

Lemma 2.10: ([12]) Let f and g are weakly compatible self-maps of a set X. If f and g have unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g

3. MAIN RESULTS

Definition 3.1: ([7]) Let P be a solid cone in a Banach space A. A sequence $\{x_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n \ll c$ for $n \geq n_0$.

Proposition 3.2:([7]) Let P be a solid cone in a Banach space A. A sequence $\{x_n\}$ and $\{y_n\}$ be a sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.

Proposition 3.3:([7]) Let P be a solid cone in Banach algebra A. A sequence $\{x_n\}$ be a sequencein P. Then the following conditions are equivalents:

(1) $\{x_n\}$ a c-sequence.

(2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n \prec c \; for \; n \geq n_0$.

(3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $x_n \preccurlyeq c \; for \; n \geq n_1$.

Proposition 3.4: ([5]) Let P be a solid cone in a Banach algebra A. A sequence $\{x_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is a c-sequence in P. Then $\{kx_n\}$ is a c-sequence.

Proposition 3.5 :([5]) Let A be a Banach algebra A with unity e, P be a cone in A and $\preccurlyeq$ be semi-order generated by the cone P. The following assertion hold true.

(1) For any $x, y \in A, a \in P$ with $x \preccurlyeq y$, we have $ax \preccurlyeq ay$.

(2) For any sequence $\{x_n\}$, $\{y_n\} \subset A \; with \; x_n \to x \; (n \to \infty) \; and \; y_n \to y (n \to \infty)$ where $x, y \in A$, we have $x_n y_n \to xy \; (n \to \infty)$.

Proposition 3.6: ([5]) Let (X, d) be complete cone metric space over a Banach algebra A and let P be the underlying solid cone in Banach algebra A. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have:

(1) $\{d(x_n, x)\}$ is a c-sequence.

(2) For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$, is a c-sequence.

Now we present generalized expanding mapping defined in ([11]).

Definition 3.7 : Let (X, d) be a cone metric space over Banach algebra A and $T : X \to X$. Then T is called a generalized expanding mapping, if for every $x, y \in X$ there exists $k \in P$ with $\mathrm{r}(k) > 1$ such that $d(Tx, Ty) \succcurlyeq kd(x, y)$.

Lemma 3.8: ([11]) Let A be a Banach algebra A with unity e, P be a cone in A if x is invertible such that $r(x) > 1$ then $r(x^{-1}) < 1$

Now we shall prove coincidence point and common fixed point theorem of generalized expanding mapping in the setting of cone metric space over Banach algebra.

Theorem 3.9: Let (X, d) be a complete cone metric space over Banach algebra A and P be the solid cone in A. Let $k_i \in P$, $(i = 1,2, \ldots 5)$ be generalized Lipschitz constants with $r\left(\sum_{i=1}^{3} k_i\right) > 1$, $\left(k_1 + \sum_{i=1}^{5} k_i\right)$ is invertible $r(k_2) \leq r(e + k_4)$, $r(k_3) \leq r(e + k_5)$, each k_i commutes with k_j for $i \neq j$ $(i = 1,2, \dots 5)$ and with $\left(k_1 + \sum_{i=1}^{5} k_i\right)^{-1}$. Suppose the mappings $f, g: X \to X$ be self-mappings and satisfies the condition

$$d(fx, fy) \succcurlyeq k_1 d(gx, gy) + k_2\, d(fx, gx) + k_3 d(fy, gy) + k_4\, d(fy, gx) + k_5 d(fx, gy) \quad (1)$$

for all $x, y \in X$, If the range of $f(X)$ contains $g(X)$ and $f(X) or\ g(X)$ is complete subspace then $f\ and\ g$ have unique point of coincidence in X. Moreover $f\ and\ g$ are weakly compatible then $f\ and\ g$ have a unique common fixed point.

Proof: Since $f\ and\ g$ are onto self-mappings so for each $x_0 \in X$, there exists $x_1 \in X$ such that $fx_1 = gx_0$, $fx_2 = gx_1, fx_3 = gx_2, \dots, fx_{n+1} = gx_n$ for $n \in \mathbb{N} \cup \{0\}$.

Now, $d(gx_n, gx_{n-1}) = d(fx_{n+1}, fx_n)$

$$\begin{aligned} &\succcurlyeq k_1 d(gx_{n+1}, gx_n) + k_2 d(fx_{n+1}, gx_{n+1}) \\ &\quad + k_3 d(fx_n, gx_n) + k_4 d(fx_n, gx_{n+1}) + k_5 d(fx_{n+1}, gx_n) \\ &= k_1 d(gx_{n+1}, gx_n) + k_2 d(gx_n, gx_{n+1}) \\ &\quad + k_3 d(gx_{n-1}, gx_n) + k_4 d(gx_{n-1}, gx_{n+1}) + k_5 d(gx_n, gx_n) \end{aligned}$$

Since $d(gx_{n-1}, gx_{n+1}) \succcurlyeq d(gx_{n+1}, gx_n) - d(gx_{n-1}, gx_n)$ (2)

$$(e - k_3) d(gx_n, gx_{n-1}) \succcurlyeq (k_1 + k_2)\, d(gx_n, xg_{n+1}) + k_4[d(gx_{n+1}, gx_n) - d(gx_{n-1}, gx_n)]$$

$$(e - k_3 + k_4) d(gx_n, gx_{n-1}) \succcurlyeq (k_1 + k_2 + k_4)\, d(gx_n, gx_{n+1}) \quad (3)$$

Also,

$$\begin{aligned} d(gx_{n-1}, gx_n) &= d(fx_n, fx_{n+1}) \\ &\succcurlyeq k_1 d(gx_n, gx_{n+1}) + k_2 d(fx_n, gx_n) \\ &\quad + k_3 d(fx_{n+1}, gx_{n+1}) + k_4 d(fx_{n+1}, gx_n) + k_5 d(fx_n, gx_{n+1}) \\ &\succcurlyeq k_1 d(gx_n, gx_{n+1}) + k_2 d(gx_{n-1}, gx_n) \\ &\quad + k_3 d(gx_n, gx_{n+1}) + k_4 d(gx_n, gx_n) + k_5 d(gx_{n-1}, gx_{n+1}) \end{aligned}$$

Using (2),

$$d(gx_{n-1}, gx_n) \succcurlyeq k_1 d(gx_n, gx_{n+1}) + k_2 d(gx_{n-1}, gx_n) + k_3 d(gx_n, gx_{n+1}) + k_5[d(gx_{n+1}, gx_n) - d(gx_{n-1}, gx_n)]$$

$$(e - k_2 + k_5) d(gx_{n-1}, gx_n) \succcurlyeq (k_1 + k_3 + k_5) d(gx_n, gx_{n+1}) \quad (4)$$

Adding (3) and (4), we get,

$$(2e - k_3 + k_5 - k_2 + k_4) d(gx_{n-1}, gx_n) \succcurlyeq (2k_1 + k_2 + k_3 + k_4 + k_5) d(gx_n, gx_{n+1})$$

$$d(gx_n, gx_{n+1}) \preccurlyeq (2e - k_3 + k_5 - k_2 + k_4)(2k_1 + k_2 + k_3 + k_4 + k_5)^{-1} d(gx_{n-1}, gx_n)$$

Let $k = k_1 + k_2 + k_3 + k_4 + k_5$

Since $r(k) > r(k_1 + k_2 + k_3) > 1$ so by Lemma 3.8 $r(k_1 + k)^{-1} < 1$

Also, $r(k_3) \leq r(e + k_5)$ So, $0 \leq r(e) + r(k_5) - r(k_3) < r(k)$

$r(k_2) \leq r(e + k_4)$ So, $0 \leq r(e) + r(k_4) - r(k_2) < r(k)$

And $r(2e - k_3 + k_5 - k_2 + k_4) < r(k_1 + k)$

Let $h = (2e - k_3 + k_5 - k_2 + k_4)(k_1 + k)^{-1}$, So $r(h) \in [0,1)$

$d(gx_n, gx_{n+1}) \preccurlyeq h\, d(gx_{n-1}, gx_n)$

Therefore, form this we get $d(gx_n, gx_{n+1}) \preccurlyeq h^n d\,(gx_0, gx_1)$

Now for $m > n$ we have by triangle inequality

$$
\begin{aligned}
d(gx_n, gx_m) &\preccurlyeq d\,(gx_n, gx_{n+1}) + d\left(gx_{n+1,}gx_{n+2}\right) + \cdots + d\left(gx_{m-1,}gx_m\right)\\
&\preccurlyeq (h^n + h^{n+1} + \cdots + h^{m-1}) d\,(gx_0, gx_1)\\
&= h^n(e + h + h^2 + h^3 + \cdots + h^{m-n-1})\, d\,(gx_0, gx_1\,)\\
&\preccurlyeq \left(\sum_{i=0}^{\infty} h^i\right) h^n d(gx_0, gx_1)\\
&\preccurlyeq h^n(e-h)^{-1} d(gx_0, gx_1).
\end{aligned}
$$

By Remark 2.8 and Lemma 2.7 it follows that, for $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$, we have

$$d(gx_n, gx_m) \preccurlyeq h^n(e-h)^{-1} d(gx_0, gx_1) \ll c.$$

Which implies that $\{gx_n\}$ is a Cauchy sequence.

First, suppose that $g(X)$ is complete, then there exists $q \in g(X) \subseteq f(X)$ such that $g(x_n) \to q$ and also $f(x_n) \to q$.

If $f(X)$ is complete, then there exists $q \in f(X)$ such that $fx_{n+1} = g(x_n) \to q,\ (n \to \infty)$. Consequently, we can find a $r \in f(X)$ such that$fr = q$. Now we show that $gr = q$

Substituting $x = r,\ \ y = x_{n+1}$ in (1), we get

$$
\begin{aligned}
d(q, gx_n) &= d(fr, fx_{n+1})\\
&\succcurlyeq k_1 d(gr, gx_{n+1}) + k_2 d(fr, gr) + k_3 d(fx_{n+1}, gx_{n+1}) + k_4 d(\,fx_{n+1}, gr)\\
&\quad + k_5 d(fr, gx_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
d(q, gx_n) &\succcurlyeq k_1 d(gr, gx_{n+1}) + k_2 d(q, gr) + k_3 d(gx_n, gx_{n+1}) + k_4 d(gx_n\,, gr)\\
&\quad + k_5 d(q, gx_{n+1})
\end{aligned}
$$

By Triangle inequality, we have

$$d(q, gx_n) \preccurlyeq d(q, gx_{n+1}) + d(gx_{n+1}, gx_n) \tag{5}$$

Also $d(gx_{n+1}, gr) \preccurlyeq d(gx_{n+1}, q) + d(q, gr)$

$$d(q, gr) \succcurlyeq d(gx_{n+1}, gr) - d(gx_{n+1}, q) \tag{6}$$

$$d(gx_n, gr) \succcurlyeq d(gx_{n+1}, gr) - d(gx_{n+1}, gx_n) \tag{7}$$

Using equation (5), (6) and (7) in above equation, we have

$$
\begin{aligned}
d(q, gx_{n+1}) + d(gx_{n+1}, gx_n) &\succcurlyeq k_1 d(gr, gx_{n+1}) + k_2\big(d(gx_{n+1}, gr\,) - d(gx_{n+1}, q)\big)\\
&\quad + k_3 d(gx_n, gx_{n+1}) + k_4(d(gx_{n+1}, gr) - d(gx_{n+1}, gx_n)) + k_5 d(q, gx_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
&(e + k_2 - k_5) d(q, gx_{n+1}) + (e - k_3 + k_4) d(gx_{n+1}, gx_n)\\
&\qquad \succcurlyeq (k_1 + k_2 + k_4) d(gx_{n+1}, gr)
\end{aligned} \tag{8}
$$

Substituting $x = x_{n+1},\ y = r$ in (1), we get

$$
\begin{aligned}
d(gx_n, q) &= d(fx_{n+1}, fr)\\
&\succcurlyeq k_1 d(gx_{n+1}, gr) + k_2 d(fx_{n+1}, gx_{n+1}) + k_3 d(fr, gr) + k_4 d(fr, gx_{n+1}) +\\
&\quad k_5 d(fx_{n+1}, gr)
\end{aligned}
$$

$$d(gx_n, q) \succcurlyeq k_1 d(gx_{n+1}, g\,r) + k_2 d(gx_n, gx_{n+1}) + k_3 d(q, gr) + k_4 d(q, gx_{n+1}) + k_5 d(gx_n, gr)$$

Using equations (5), (6) and (7) in above equation, we have

$$
\begin{aligned}
d(q, gx_{n+1}) + d(gx_{n+1}, gx_n) &\succcurlyeq k_1 d(gx_{n+1}, gr) + k_2 d(gx_n, gx_{n+1})\\
&\quad + k_3\big(d(gx_{n+1}, gr\,) - d(gx_{n+1}, q)\big) + k_4 d(q, gx_{n+1}) + k_5(d(gx_{n+1}, gr) - d(gx_{n+1}, gx_n))
\end{aligned}
$$

$$(e + k_3 - k_4) d(gx_{n+1}, q) + (e - k_2 + k_5) d(gx_n, gx_{n+1})$$

$$\succcurlyeq (k_1 + k_3 + k_5) d(gx_{n+1}, gr) \quad (9)$$

Adding (8) and (9), we get

$$(2e + k_2 - k_4 + k_3 - k_5) d(q, gx_{n+1}) + (2e - k_3 + k_5 - k_2 + k_4) d(gx_{n+1}, gx_n)$$
$$\succcurlyeq (2k_1 + k_2 + k_3 + k_4 + k_5) d(gx_{n+1}, gr)$$
$$d(gx_{n+1}, gr) \preccurlyeq (2e + k_2 - k_4 + k_3 - k_5)\,(2k_1 + k_2 + k_3 + k_4 + k_5)^{-1}\ d(q, gx_{n+1})$$
$$+(2e - k_3 + k_5 - k_2 + k_4)(2k_1 + k_2 + k_3 + k_4 + k_5)^{-1} d(gx_{n+1}, gx_n)$$
$$d(gx_{n+1}, gr) \preccurlyeq (2e + k_2 - k_4 + k_3 - k_5)\,(k_1 + k)^{-1}\ d(q, gx_{n+1}) + hd(gx_{n+1}, gx_n)$$

Let $l = (2e + k_2 - k_4 + k_3 - k_5)\,(k_1 + k)^{-1}$

$$d(gx_{n+1}, gr) \preccurlyeq l\, d(q, gx_{n+1}) + h\, d(gx_{n+1}, gx_n)$$

From the Definition 3.1 Proposition 3.2, 3.3, 3.4 and 3.6 we have $d(gr, gx_{n+1}) \preccurlyeq u_n$ where $u_n = l\, d(q, gx_{n+1}) + h\, d(gx_{n+1}, gx_n)$ is a c-sequence in cone P. Hence, for each $c \gg \theta$ we have $d(gr, gx_{n+1}) \ll c$, so by Lemma 2.6 $d(gr, gx_{n+1}) = \theta$

So $gx_{n+1} \to gr\ (n \to \infty)$ From Lemma 2.9 we have $gr = q$.

Hence $fr = gr = q$, so f and g have point of coincidence.

Now we show that f and $g$have unique point of coincidence.

If there exists another coincidence point r^* such that $fr^* = gr^*$ then we have,

$$d(gr^*, gr) = d(fr^*, fr)$$
$$\succcurlyeq k_1 d(gr^*, gr) + k_2 d(fr^*, gr^*) + k_3 d(fr, gr) + k_4 d(gr^*, fr) + k_5 d(gr, fr^*)$$
$$\succcurlyeq k_1 d(gr^*, gr) + k_2 d(gr^*, gr^*) + k_3 d(gr, gr) + k_4 d(gr^*, gr)$$
$$+ k_5 d(gr, gr^*)$$
$$\succcurlyeq k_1 d(gr^*, gr) + k_4 d(gr^*, gr) + k_5 d(gr, gr^*)$$
$$d(gr^*, gr) \succcurlyeq (k_1 + k_4 + k_5) d(gr^*, gr)$$
$$(k_1 + k_4 + k_5 - e) d(gr^*, gr) \preccurlyeq \theta$$
$$d(gr^*, gr) = \theta$$

So, $gr^* = gr$

Thus $f\ and\ g$ have unique point of coincidence.

If $f\ and\ g$ are weakly compatible, then by using Lemma 2.10 we claim that f and g have a unique common fixed point. ■

Corollary 3.10: Let (X, d) be a complete cone metric space over Banach algebra A and let the mapping $f: X \to X$ be onto and satisfies the condition $d(fx, fy) \succcurlyeq k\, d(x, y)$ for all $x, y \in X$, and $k \in P, k$ is invertible where $r(k) > 1$. Then, f has a unique fixed point in X.

Corollary 3.11: Let (X, d) be a complete cone metric space over Banach algebra A and let the mapping $f: X \to X$ be onto and satisfies the condition $d(fx, fy) \succcurlyeq k_1\, d(fx, x) + k_2 d(fy, y)$ for all $x, y \in X,\ and\ k_1, k_2 \in P, (k_1 + k_2)$ is invertible where $r(k_1 + k_2) > 1$. Then, f has a unique fixed point in X.

Corollary 3.12: Let (X, d) be a complete cone metric space over Banach algebra A and let the mapping $f: X \to X$ be onto and satisfies the condition

$$d(fx, fy) \succcurlyeq k_1 d(x, y) + k_2\, d(fx, x) + k_3 d(fy, y) + k_4\, d(fx, y) + k_5 d(fy, x)$$

for all $x, y \in X,\ and\ k_i \in P, (for\ i = 1,2, \dots 5)$ be generalized Lipschitz constants .
Where each $k_\mathrm{i}$commuts with k_j for $i \neq j$, and with $(k_1 + k_2 + k_3)^{-1}$, $r(k_1 + k_2 + k_5) >$ 1 and $r(k_2) \leq r(e + k_4)$, $r(k_3) \leq r(e + k_5)$, $(k_1 + k_2 + k_3)$ is invertible
Then, f has a unique fixed point in X.

Corollary 3.13: Let (X, d) be a complete cone metric space over Banach algebra A and P be the solid cone in A. Let $k \in P$, be generalized Lipschitz constants with $r(k) > 1$,
k is invertible. Suppose the mappings $f, g: X \to X$ be self-mappings and satisfies the condition $d(fx, fy) \succcurlyeq kd(gx, gy)$ for all $x, y \in X$, If the range of $f(X)$ contains $g(X)$ and $f(X)$ or $g(X)$ is complete then $f\ and\ g$ have unique point of coincidence in X. Moreover $f\ and\ g$ are weakly compatible then $f\ and\ g$ have a unique common fixed point.

Corollary 3.14: Let (X, d) be a complete cone metric space over Banach algebra A and P be the solid cone in A. Let $k_i \in P$, $(\ i = 1,2,3)$ be generalized Lipschitz constants with $r(k_1 + k_2 + k_3) > 1$, $(k_1 + k_2 + k_3\ )$ is invertible $r(k_2) \leq r(e + k_3)$, each k_i commutes with k_j for $i \neq j$ $(\ i = 1,2,3)$ and with $(k_1 + k_2)^{-1}$. Suppose $f, g: X \to X$ be self-mappings and satisfies the condition

$$d(fx, fy) \succcurlyeq k_1 d(gx, gy) + k_2\ [d(fx, gx) + d(fy, gy)] + k_3\ [d(gx, fy) + d(gy, fx)]$$

for all $x, y \in X$, If the range of $f(X)$ contains $g(X)$ and $f(X)$ or $g(X)$ is complete subspace then $f\ and\ g$ have unique point of coincidence in X. Moreover $f\ and\ g$ are weakly compatible then $f\ and\ g$ have a unique common fixed point.

Remark 3.15: Theorem 3.9 is a generalization and extension of theorems of [16, 9]
Now we present an example in support of Theorem 3.9.

Example 3.16: In Example 2.4 let $f, g: X \to X$ be self-mappings defined by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $g(1) = 1$, $g(2) = 2$, $g(3) = 3$. Let $k_i \in P$ $(i = 1,2, \dots, 5)$
defined with

$k_1(t) = \frac{t}{8} + \frac{1}{8}$, $k_2(t) = \frac{t}{5} + \frac{1}{5}$, $k_3(t) = \frac{t}{6} + \frac{1}{6}$, $k_4(t) = \frac{t}{60} + \frac{1}{60}$ and $k_5(t) = \frac{t}{60} + \frac{1}{60}$.

We can see that all the conditions of Theorem 3.9 are satisfied. the point x=1 is the unique coincidence and common fixed point of f and g.

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