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Fixed Point Theorem and Common Fixed Point Theorem in Cone Metric Spaces

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ABSTRACT

The aim of this paper is to prove a fixed point theorem and a common fixed point theorem for Ciric-type contraction mapping in the setting of Cone metric spaces. Our results extends and generalizes the results of Ciric in[5], Rezapour[14] and Turkoglu *et al.*[18].

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1. INTRODUCTION

In 1922, the Polish mathematician, Stephen Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This result is called Banach's fixed point theorem. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. After that, many authors have extended, generalized and improved Banach's fixed point theorem in different ways and different spaces.

In 2007, Huang and Zhang[9] introduced the notion of Cone metric space and described there convergence in cone metric spaces. They also proved some fixed point theorems in cone metric spaces. Turkoglu and Abuloha[17] generalized some definitions and topological concepts of cone metric spaces and proved there that every cone metric space is a topological space. They also generalized the concept of diametrically contractive mappings and proved some fixed point theorems in cone metric spaces. In[1], the authors proved some fixed point theorems in cone metric spaces which generalized those in[9]. In[11], the authors defined the quasi-contraction on cone metric spaces and they proved some fixed point theorems. For more recent fixed point theorems in cone metric spaces we refer to (see[2,3,8,12,13,15,16,20]).

On the other hand, generalized contraction mappings, introduced in[4], are of great importance in fixed point theory. After that and in the last decade, in[7], J. Gornicki, B. E. Rhoades used generalized contraction mappings to obtain common fixed point theorems.

In this paper, we to prove a fixed point theorem and a common fixed point theorem for Ciric-type contraction mapping in the setting of Cone metric spaces. Our results extends and generalizes the results of Ciric in[5] and Turkoglu *et al*.[18].

2. PRELIMINARIES

Let E be a real Banach space and P a subset of E. Then, P is called a cone if and only if

P1) P is closed, non-empty and $P \neq \{0\}$,

P2) $a,b \in R$; $a,b \geq 0$; $x,y \in P \Rightarrow ax+by \in P$,

P3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only $y-x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in IntP$. Here $IntP$ denotes the interior of P.

The cone P is called normal if there is a number K, such that for all $x,y \in E, 0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$, where K is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq y$ for some $y \in E$. Then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is called regular if every decreasing sequence which is bounded below is convergent [14]. P is called a minihedral cone if $\sup\{x,y\}$ exists for all $x,y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum and hence any subset of E which is bounded from below has an infimum[6]. Throughout this paper we assume that the cone P is normal with constant K and P is a cone in E with $\operatorname{int} P \neq ¤$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 (see[9]). A cone metric space is an ordered pair (X,d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:

D1) $0 < d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$, D2) $d(x,y)=d(y,x)$ for all $x,y\in X$,

D3) $d(x,y)\le d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Definition 2.2 (see[9]). Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x\in X$. If for any $c\in E$ with $c\gg 0$, there is N such that for all $n>N$,$d(x_n,x)\ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x. (i.e. $\lim_{n\to\infty} x_n = x$ or $x_n\to x$ as $n\to\infty$).

Definition 2.3 (see[9]). Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X , if for any $c\in E$ with $c\gg 0$, there is N such that for all $n,m>N$, $d(x_n,x_m)\ll c$, then $\{x_n\}$ is called a Cauchy sequence in X .

Lemma 2.1 (see[9]). Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $y_n\to y, x_n\to x$ as $n\to\infty$, then $d(x_n,y_n)\to d(x,y)$ as $n\to\infty$.

Lemma 2.2 (see[9]). Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $d(x_n,x)\to 0$ as $n\to\infty$.

Lemma 2.3 (see[9]). Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ Cauchy sequence if and only if $d(x_n,x_m)\to 0$ as $m,n\to\infty$.

3. MAIN RESULTS

For $x_1,x_2\in X$ the scalar distant $d_c(x_1,x_2)$ between x_1 and x_2 is defined by $d_c(x_1,x_2)=\|d(x_1,x_2)\|$.

Theorem 3.1. Let (X,d) be a complete cone metric space with a normal constant $K\ge 1$ and $T:X\to X$ a self-map on X such that for each $x,y\in X$:

$$d_c(Tx,Ty)\le a(x,y)\max\left\{d_c(x,y),d_c(x,Tx),d_c(y,Ty),\frac{1}{2}\left[d_c(x,Ty)+d_c(y,Tx)\right]\right\} \\ +b(x,y)\max\{d_c(x,Tx),d_c(y,Ty)\}+c(x,y)\left[d_c(x,Ty)+d_c(y,Tx)\right] \tag{3.1}$$

where a,b,c are functions from $X\times X$ into [0,1) such that

$$\alpha = \sup\left\{Ka(x,y)+b(x,y)+2Kc(x,y): x,y\in X\right\}<1, \tag{3.2}$$

then

(i) T has a unique fixed point , say $u\in X$,

(ii) $T^n x\to u$ as $n\to\infty$, for each $x\in X$,

(iii) $d_c\left(T^n x,u\right)\le \dfrac{\alpha^n}{1-\alpha} d_c\left(x,Tx\right)$.

Proof. Fix $x\in X$. Let $\left\{x_n\right\}$ be defined by $x_0=x,\ x_1=Tx_0,\ x_2=Tx_1,\cdots,x_{n+1}=Tx_n,\cdots$. From (2.1),

$$\begin{aligned}
d_c\left(x_n,x_{n+1}\right)&=d_c\left(Tx_{n-1},Tx_n\right)\\
&\le a\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_{n-1},Tx_{n-1}\right),d_c\left(x_n,Tx_n\right),\frac{1}{2}\left[d_c\left(x_{n-1},Tx_n\right)+d_c\left(x_n,Tx_{n-1}\right)\right]\right\}\\
&\quad+b\max\left\{d_c\left(x_{n-1},Tx_{n-1}\right),d_c\left(x_n,Tx_n\right)\right\}+c\left[d_c\left(x_{n-1},Tx_n\right)+d_c\left(x_n,Tx_{n-1}\right)\right]\\
&\le a\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right),\frac{1}{2}\left[d_c\left(x_{n-1},x_{n+1}\right)+d_c\left(x_n,x_n\right)\right]\right\}\\
&\quad+b\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right)\right\}+c\left[d_c\left(x_{n-1},x_{n+1}\right)+d_c\left(x_n,x_n\right)\right]
\end{aligned}$$

Or

$$\begin{aligned}
d_c\left(x_n,x_{n+1}\right)&\le a\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right),\frac{1}{2}d_c\left(x_{n-1},x_{n+1}\right)\right\}\\
&\quad+b\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right)\right\}+cd_c\left(x_{n-1},x_{n+1}\right)
\end{aligned} \tag{3.3}$$

where a,b,c are evaluate at $\left(x_{n-1},x_n\right)$.

Now by Triangular inequality, we have

$$d\left(x_{n-1},x_{n+1}\right)\le d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n+1}\right)$$

Hence

$$\begin{aligned}
d_c\left(x_{n-1},x_{n+1}\right)&\le K\left\|d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n+1}\right)\right\|\\
&\le K\left[d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n+1}\right)\right]
\end{aligned}$$

$$\le 2K\max\left\{d\left(x_{n-1},x_n\right),d\left(x_n,x_{n+1}\right)\right\} \tag{3.4}$$

By (3.4) equation (3.3) turn to be

$$d_c\left(x_n,x_{n+1}\right)\le\left(Ka+b\right)\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right)\right\}+2Kc\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right)\right\}$$

Then

$$d_c\left(x_n,x_{n+1}\right)\le\alpha\max\left\{d_c\left(x_{n-1},x_n\right),d_c\left(x_n,x_{n+1}\right)\right\}$$

Since $\alpha<1$, we have

$$d_c(x_n, x_{n+1}) \le \mathsf{a}\, d_c(x_{n-1}, x_n) \tag{3.5}$$

Proceeding in this way, we obtain

$$\begin{aligned} d_c(x_n, x_{n+1}) &\le \mathsf{a}\, d_c(x_{n-1}, x_n) \\ &\le \mathsf{a}^2 d_c(x_{n-2}, x_{n-1}) \\ &\le \\ &\vdots \\ &\le \mathsf{a}^n d_c(x, Tx) \end{aligned} \tag{3.6}$$

Now we will prove that $\{x_n\}$ is a Cauchy sequence.

For this using normality of cone, equation (3.6) and that $\|\cdot\|$ satisfies the triangular inequality, we obtain

$$\begin{aligned} d_c(x_n, x_m) &\le K\left[\|d_c(x_n, x_{n+1})\| + \|d_c(x_{n+1}, x_{n+2})\| + \cdots + \|d_c(x_{m-1}, x_m)\|\right] \\ &\le K\left[\|d(x_n, x_{n+1})\| + \|d(x_{n+1}, x_{n+2})\| + \cdots + \|d(x_{m-1}, x_m)\|\right] \\ &\le K\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)\right] \\ &\le K\left[\mathsf{a}^n d_c(x, Tx) + \mathsf{a}^{n+1} d_c(x, Tx) + \cdots + \mathsf{a}^{m-1} d_c(x, Tx)\right] \\ &\le \frac{\mathsf{a}^n}{1-\mathsf{a}} K d_c(x, Tx) \end{aligned}$$

Or

$$d_c(x_n, x_m) \le \frac{\mathsf{a}^n}{1-\mathsf{a}} K d_c(x, Tx) \tag{3.7}$$

Letting limit as $m, n \to \infty$ in (3.7), Lemma 2.2 implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, then there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \tag{3.8}$$

Next, we will show that u is a fixed point of T .
From (3.1) and (3.2)

$$\begin{aligned} d_c(Tu, Tx_n) &\le a \max\left\{d_c(u, x_n), d_c(u, Tu), d_c(x_n, Tx_n), \frac{1}{2}\left[d_c(u, Tx_n) + d_c(x_n, Tu)\right]\right\} \\ &\le b \max\left\{d_c(u, Tu), d_c(x_n, Tx_n)\right\} + c\left[d_c(u, Tx_n) + d_c(x_n, Tu)\right] \\ &\le (a+b+2c) \max\left\{d_c(u, x_n), d_c(u, Tu), d_c(x_n, x_{n+1}), d_c(u, x_{n+1}), d_c(x_n, Tu)\right\} \\ &\le \mathsf{a} \max\left\{d_c(u, x_n), d_c(u, Tu), d_c(x_n, x_{n+1}), d_c(u, x_{n+1}), d_c(x_n, Tu)\right\} \end{aligned} \tag{3.8}$$

Letting $n \to \infty$, then by (3.8) and Lemma 2.1, we have

$$d_c(u,Tu) \le \lambda d_c(u,Tu) \tag{3.9}$$

Since $\lambda < 1$, then $d_c(u,Tu) = 0$. Hence $\|d(Tu,u)\| = 0$ implies $Tu = u$.

To prove the uniqueness of fixed point, assume $x, y \in X$ and $x \neq y$ are two fixed points of T. Then from (3.1),

$$\begin{aligned} d_c(x,y) &= d_c(Tx,Ty) \\ &\le a \max\left\{ d_c(x,y), d_c(x,Tx), d_c(y,Ty), \frac{1}{2}\left[d_c(x,Ty) + d_c(y,Tx) \right] \right\} \\ &\quad + b \max\left\{ d_c(x,Tx), d_c(y,Ty) \right\} + c\left[d_c(x,Ty) + d_c(y,Tx) \right] \\ &\le (a+2c) d_c(x,y) \le \lambda d_c(x,y) \end{aligned}$$

Since $\lambda < 1$, then $d_c(x,y) = 0$ which implies $x = y$. Since $x \in X$ be arbitrary then from (3.8), we conclude that (ii) holds.

To show (iii), taking the limit in (3.7) as $n \to \infty$ and making use of Lemma 2.1, we get $d_c(T^n x, u) \le \dfrac{\lambda^n}{1-\lambda} d_c(x,Tx)$ for each n. This completes the proof of the theorem.

If we put $b = c = 0$ in Theorem 3.1, we get the following corollary as generalization of Theorem 1 of [18] as a special case.

Corollary 3.1. Theorem 3.1. Let (X,d) be a complete cone metric space with a normal constant $K \ge 1$ and $T : X \to X$ a self-map on X such that for each $x, y \in X$:

$$d_c(Tx,Ty) \le \lambda \max\left\{ d_c(x,y), d_c(x,Tx), d_c(y,Ty), \frac{1}{2}\left[d_c(x,Ty) + d_c(y,Tx) \right] \right\}$$

where $\lambda \in (0,1)$ with $\lambda K < 1$, then

(i) T has a unique fixed point , say $u \in X$,
(ii) $T^n x \to u$ as $n \to \infty$, for each $x \in X$,
(iii) $d_c(T^n x, u) \le \dfrac{\lambda^n}{1-\lambda} d_c(x,Tx)$.

Let S be a non-empty set and let $\{T_\alpha\}_{\alpha \in J}$ be a family of self-mappings on S and J an indexing set. A point $u \in S$ is called a common fixed point for a family $\{T_\alpha\}_{\alpha \in J}$ if and only if $u = T_\alpha u$ for each T_α .

Theorem 3.2. Let (X,d) be a complete cone metric space with a normal constant $K \geq 1$ and $\{T_{\sim}\}_{\sim \in J}$ a family of self-mappings of X. If there exists a fixed $\in J$ such that for each $\sim \in J$:

$$d_c\left(T_{\sim}x, T\ y\right) \leq a\max\left\{d_c(x,y), d_c(x,T_{\sim}x), d_c(y,T\ y), \frac{1}{2}\left[d_c(x,T\ y)+d_c(y,T_{\sim}x)\right]\right\} \quad (3.10)$$
$$+b\max\left\{d_c(x,T_{\sim}x), d_c(y,T\ y)\right\}+c\left[d_c(x,T\ y)+d_c(y,T_{\sim}x)\right]$$

where a,b,c are functions from $X \times X$ into [0,1) such that

$$ª = \sup\left\{Ka(x,y)+b(x,y)+2Kc(x,y): x,y \in X\right\} < 1. \quad (3.11)$$

Then all $T_{\sim}$ have a unique common fixed point which is a unique fixed point of each $T_{\sim}, \sim \in J$.

Proof. Let $\sim \in J$ and $x \in X$ be arbitrary. Consider a sequence defined by $x_0 = x$, $x_{2n+1} = T_{\sim}x_{2n}, x_{2n+2} = T\ x_{2n+1}$, $n \geq 0$. From (3.10), we get

$$d_c(x_{2n+1},x_{2n+2}) = d_c\left(T_{\sim}x_{2n}, T\ x_{2n+1}\right)$$
$$\leq a\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n},T_{\sim}x_{2n}), d_c(x_{2n+1},T\ x_{2n+1}), \frac{1}{2}\left[d_c(x_{2n},T\ x_{2n+1})+d_c(x_{2n+1},T_{\sim}x_{2n})\right]\right\}$$
$$+b\max\left\{d_c(x_{2n},T_{\sim}x_{2n}), d_c(x_{2n+1},T\ x_{2n+1})\right\}+c\left[d_c(x_{2n},T\ x_{2n+1})+d_c(x_{2n+1},T_{\sim}x_{2n})\right]$$
$$\leq a\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2}), \frac{1}{2}\left[d_c(x_{2n},x_{2n+2})+d_c(x_{2n+1},x_{2n+1})\right]\right\}$$
$$+b\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2})\right\}+c\left[d_c(x_{2n},x_{2n+2})+d_c(x_{2n+1},x_{2n+1})\right]$$

Since

$$d(x_{2n},x_{2n+2}) \leq d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})$$
$$\left\|d(x_{2n},x_{2n+2})\right\| \leq K\left[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})\right] \quad (3.12)$$
$$d_c(x_{2n},x_{2n+2}) \leq K[d_c(x_{2n},x_{2n+1})+d_c(x_{2n+1},x_{2n+2})]$$

So,

$$\frac{1}{2}d_c(x_{2n},x_{2n+2}) \leq \frac{1}{2}K\left[d_c(x_{2n},x_{2n+1})+d_c(x_{2n+1},x_{2n+2})\right]$$
$$\leq K\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2})\right\}$$

We have,

$$d_c(x_{2n+1},x_{2n+2}) \leq aK\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2})\right\}$$
$$+b\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2})\right\}$$
$$+cK\max\left\{d_c(x_{2n},x_{2n+1}), d_c(x_{2n+1},x_{2n+2})\right\}$$

Now,

Case I. If $\max\left\{d_c\left(x_{2n},x_{2n+1}\right),d_c\left(x_{2n+1},x_{2n+2}\right)\right\}=d_c\left(x_{2n},x_{2n+1}\right)$, then

$$d_c\left(x_{2n+1},x_{2n+2}\right)\le\left[\frac{aK+b+cK}{1-cK}\right]d_c\left(x_{2n},x_{2n+1}\right)$$
$$\le\alpha\, d_c\left(x_{2n},x_{2n+1}\right)$$

Case II. If $\max\left\{d_c\left(x_{2n},x_{2n+1}\right),d_c\left(x_{2n+1},x_{2n+2}\right)\right\}=d_c\left(x_{2n+1},x_{2n+2}\right)$, then

$$d_c\left(x_{2n+1},x_{2n+2}\right)\le\left[\frac{cK}{1-(aK+b+cK)}\right]d_c\left(x_{2n},x_{2n+1}\right)$$
$$\le\alpha\, d_c\left(x_{2n},x_{2n+1}\right)$$

Similarly, we may find that $d_c\left(x_{2n},x_{2n+1}\right)\le\alpha\, d_c\left(x_{2n-1},x_{2n}\right)$.

Thus for any $n\ge1$, we have

$$d_c\left(x_n,x_{n+1}\right)\le\alpha\, d_c\left(x_{n-1},x_n\right)\le\alpha^2 d_c\left(x_{n-2},x_{n-1}\right)\le\cdots\le\alpha^n d_c\left(x_0,x_1\right).$$

Now, we show that $\left\{x_n\right\}$ is a Cauchy sequence in X .

For $m>n$, we get

$$d_c\left(x_n,x_m\right)\le K\left[d_c\left(x_n,x_{n+1}\right)+d_c\left(x_{n+1},x_{n+2}\right)+\cdots+d_c\left(x_{m-1},x_m\right)\right]$$
$$\le K\left[\alpha^n d_c\left(x_0,x_1\right)+\alpha^{n+1}d_c\left(x_0,x_1\right)+\cdots+\alpha^{m-1}d_c\left(x_0,x_1\right)\right]$$
$$\le K\left[\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1}\right]d_c\left(x_0,x_1\right)$$
$$\le K\frac{\alpha^n}{1-\alpha}d_c\left(x_0,x_1\right) \tag{3.13}$$

Letting limit as $m,n\to\infty$ and using Lemma 2.2, we conclude that $\left\{x_n\right\}$ is a Cauchy sequence. Since X is complete, there is a $z\in X$ such that

$$\lim_{n\to\infty}x_n=z \tag{3.14}$$

From (3.10), we have

$$d_c\left(T\ z,x_{2n+1}\right)=d_c\left(T\ z,T_{\sim}x_{2n}\right)$$
$$\le a\max\left\{d_c\left(z,x_{2n}\right),d_c\left(z,T\ z\right),d_c\left(x_{2n},T_{\sim}x_{2n}\right),\frac{1}{2}\left[d_c\left(z,T_{\sim}x_{2n}\right)+d_c\left(x_{2n},T\ z\right)\right]\right\}$$
$$+b\max\left\{d_c\left(z,T\ z\right),d_c\left(x_{2n},T_{\sim}x_{2n}\right)\right\}+c\left[d_c\left(z,T_{\sim}x_{2n}\right)+d_c\left(x_{2n},T\ z\right)\right]$$

Taking the limit as $n\to\infty$ and using Lemma 2.1, we get

$d_c\left(T\ z,z\right) \le \left(a+b+c\right) d_c\left(T\ z,z\right) \Rightarrow d_c\left(T\ z,z\right)=0$ as $a+b+2c<1$ and so $T\ z=z$.

Now we show that z is a fixed point of all $\{T_\sim\}_{\sim\in J}$, let $\sim \in J$ be arbitrary. Then from (3.10) with $x=y=z=T\ z$, we have

$$\begin{aligned} d_c\left(z,T_\sim z\right) &= d_c\left(T\ z,T_\sim z\right) \\ &\le a \max\left\{d_c\left(z,z\right), d_c\left(z,T\ z\right), d_c\left(z,T_\sim z\right), \frac{1}{2}\left[d_c\left(z,T\ z\right)+d_c\left(z,T\ z\right)\right]\right\} \\ &\quad + b \max\left\{d_c\left(z,T\ z\right), d_c\left(z,T_\sim z\right)\right\} + c\left[d_c\left(z,T\ z\right)+d_c\left(z,T\ z\right)\right] \\ &\le \left(a+b+c\right) d_c\left(z,T_\sim z\right) \end{aligned}$$

and hence $T_\sim z = z$. Thus all $T_\sim$ have a common fixed point.

Suppose that w is another fixed point of T . Then as proved above, w is a common fixed point of all $\{T_\sim\}_{\sim\in J}$. Thus from (3.10), we have

$d_c\left(z,w\right) = d_c\left(T\ z,T_\sim w\right) \le (a+b+c) d_c\left(z,w\right)$ and so $z=w$.

Thus z is a unique common fixed point of all $\{T_\sim\}_{\sim\in J}$.

REFERENCES

1. M. Abbas, B. E. Rhoades, Fixed and periodic point results in Cone metric spaces, *Appl. Math. Lett*., 9, 1364-1368 (2009).
2. M. Arshad, A. Azam, P. Vetro, Some common fixed point results in cone uniform spaces, *Fixed Point Theory and Appl*., Article ID 493965 (2009).
3. A. Azam, M. Arshad, Common fixed points of generalized contractive maps in cone metric spaces, *Bull. Iranian Math. Soc*., 35, 255-264 (2009).
4. LJ. B. Ciric, Generalized contraction and fixed point theorems, *Publ. Inst. Math*., 12, 19-26 (1971).
5. LJ. B. Ciric, On a family of contractive maps and fixed points, *Publ. Inst. Math*., 17, 45-51 (1974).
6. K. Deimling, Nonlinear Functional Analysis, Springer-Verlage, Berlin, Heidelberg and New York, (1985).
7. J. Gornicki, B. E. Rhoades, A general fixed point theorem for involution, *Indian J. Pure Appl. Math*., 27, 13-23 (1996).
8. R. H. Haghi, SH. Rezapour, Fixed points of multifunctions on regular cone metric spaces, *Expo. Math*., 28, 71-77 (2010).
9. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl*., 332, 1468-1476 (2007).

10. D. Ilic, V. Rekocevic, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl*., 341, 876-882 (2008).
11. D. Ilic, V. Rekocevic, Quasi-contraction on cone metric space, *Appl. Math. Lett*., 22, 728-731 (2009).
12. D. Klim, D. Wardowski, Dynamic process and fixed points of set valued nonlinear contractions in cone metric spaces, *Nonlinear Anal*., 71, 5170-5175 (2009).
13. S. Radonevic, Common fixed points under contractive conditions in cone metric spaces, *Comput. Math. Appl*., 58, 1273-1278 (2009).
14. SH. Rezapour, Best approximations in cone metric spaces, *Math. Morav*., 11, 85-88 (2007).
15. SH. Rezapour, R. H. Haghi, Fixed point of multifunctions on cone metric spaces, *Numr. Funct. Anal. Optim*., 30, 825-832 (2009).
16. SH. Rezapour, R. H. Haghi, Two results about fixed point of multifunctions, *Math. Proc. R. Ir. Azad*, 109, 155-161 (2009).
17. D. Turkoglu, M. Abuloha, Cone metric spaces and fixed point theorems in Diametrically contractive mappings, *Acta Sinica Math*., 26, 489-496 (2010).
18. D. Turkoglu, M. Abuloha, T. Abdeljawad, Fixed points of generalized contraction mappings in cone metric spaces, *Math. Commun*., 16, 325-334 (2011).
19. P. Vetro, Common fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo*, 56, 464-468 (2007).
20. D. Wardowski, Endpoint and fixed points of set valued contractions in cone metric spaces, *Nonlinear Anal*., 71, 2259-2261 (2009).