Journal of Computer and Mathematical Sciences, *Vol.8(3), 114-121 March 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Generalization of Rakotch's Fixed Point Theorem

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(Received on: March 30, Accepted: March 31, 2017)

ABSTRACT

In this paper we get some generalizations of Rakotch's results[10] using the notion of ω −distance on a metric space.

Mathematics Subject Classification: 47H10, 54E50.

Keywords: fixed point, completeness, ω −Rakotch contraction.

1. INTRODUCTION

In 1996, O. Kada, T. Suzuki & W. Takahashi[6] introduced the concept of ω −distance on a metric space, gave some examples, properties of ω −distance and they improved Caristi's fixed point[1], Ekeland's " variational principle[5] and the non-convex minimization theorem according to W. Takahashi[17]. Finally, by the use of the concept of ω −distance they proved a fixed point theorem in a complete metric space. This theorem generalized the fixed theorems of Subrahmanyan[14], Kannan[7] and Ciric[3]. In the same year T. Suzuki & W. Takahashi[15] gave another property of the !-distance and using this notion they proved a fixed point theorem for set-valued mappings on complete metric spaces which are related with Nadler's fixed point theorem[9] and Edelstein theorem[4]. Moreover, they gave a characterization of completeness metric spaces. In 1997, T. Suzuki[16], proved several fixed point theorems which are generalizations of the Banach contraction principle and Kannan's fixed point theorems, and moreover, they discuss a characterization of metric completeness. In this paper we prove some fixed point theorems which are generalizations of Rakotch's theorem.

2. PRELIMINARIES

Throughout this paper we denote by N the set of positive integers, by R the set of real numbers and $R^+ = [0, +\infty]$.

Definition 2.1. Let (M, d) be a metric space. A function $p : M \times M \to [0, +\infty]$ is called a ω −distance on M if the following conditions are satisfied:
$\omega_1. - p(x, z) < p(x, y) + p(y, z)$ for any $x, y, z \in M$.
$\omega_2.$ −For any $x \in M, p(x, \cdot) : M \to [0, +\infty]$ is lower semi continuous.
$\omega_3.$ −For any $\epsilon > 0$ exists $\delta = \delta(\epsilon) > 0$ such that,
$p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The metric d is a ω −distance on M. Some other examples of ω −distances are given in[6] and[15]. The following results are crucial in the proof of our theorems. The next lemma was proved in[6].

Lemma 2.2. Let (M, d) be a metric space and let p be a ω −distance on M. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in M$. Then the following hold:
$a. -$ If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$ then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$ then $y = z$.
$b. -$ If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$ then $\{y_n\}$ converges to z.
$c. -$ If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$ then $\{x_n\}$ is a Cauchy sequence.
$d. -$ If $p(y, x_n) \leq \alpha_n$ for any $n \in N$ then $\{x_n\}$ is a Cauchy sequence.

Definition 2.3. Let (M, d) be a metric space. A finite sequence $\{x_0, x_1, \ldots, x_n\}$ of points of M is called an ϵ-chain joining x_0 and x_n if $d(x_{i-1}, x_i) < \epsilon$ for each $\epsilon > 0, i = 1, 2, \ldots, n$.

Definition 2.4. A metric space (M,d) is said to be ϵ-chainable if for each pair (x, y) of its points there exists an ϵ-chain joining x and y.
Every connected metric space is ϵ -chainable but the converse in not always true. However, for compact spaces both are equivalent. The following result was proved in[15].

Lemma 2.5. Let $\epsilon \in (0, +\infty)$ and let (M, d) be an ϵ-chainable metric space. Then the function $p : M \times M \to [0, +\infty)$ defined by
$p(x, y) = inf\{ \sum_{i=1}^{n} d(x_{i-1}, x_i) / \{x_0, x_1, \ldots, x_n\}$ is an ϵ-chain joining x and y is a ω −distance on M.
We extend the class of functions introduced by Rakotch [10] in the following definition.

Definition 2.6. Let (M, d) be a metric space and let p be a ω −distance on M. We denote by $\mathcal{F}$ the family of functions $\lambda(x, y)$ satisfying the following conditions:
$a. - \lambda(x, y) = \lambda(p(x, y)), i.e., \lambda$ is dependent on the ω − distance p on M.
$b. - 0 \leq \lambda(p) < 1$ for every $p > 0$.
$c. - \lambda(p)$ is monotonically decreasing function of p.
Now we introduce the following definition.

Definition 2.7. Let (M, d) be a metric space and let p be a ω −-distance on M. A mapping $T : M \to M$ is called a ω −Rakotch contraction if there exists a function $\lambda(x, y) \in \mathcal{F}$ such that
$p(Tx, Ty) \leq \lambda(x, y) p(x, y)$
for all $x, y \in M$.

Remarks:
$a.-$ If $p = d$ then T is called a Rakotch contraction.
$b.-$ If $\lambda(x,y) = k, 0 \leq k < 1$ then we get for all $x, y \in M$
$p(Tx,Ty) \leq kp(x,y)$.
T is called an ω −contraction[6] and[15], and if $p = d$ then T is a Banach contraction.
c.- If $\lambda(x,y) = k$, $0 \leq k < 1$ then for all $x \neq y$ implies
$p(Tx,Ty) < p(x,y)$
and we call T a ω −contractive mapping. It is clear that if $p = d$ then $x \neq y$ implies
$d(Tx,Ty) < d(x,y)$ and T is called a contractive mapping.

3. FIXED POINT THEOREMS

The next result generalizes Rakotch's theorem[10].

Theorem 3.1. Let (M,d) be a complete metric space and let p be an ω −distance on M. Let $T : M \to M$ be an ω −Rakotch contraction. Then there exists a unique $z \in M$ such that $Tz = z$. Further, the z satisfies $p(z,z) = 0$.

Proof: *Since* T is a ω −Rakotch contraction there exists a mapping $\lambda(x,y) \in F$ such that
$p(Tx,Ty) \leq \lambda(x,y)\, max\,\{p(x,y), p(x,Tx), p(y,Ty)\}$
for all $x, y \in M$.
Let $x_0 \in M$ and define $x_n = T^n x_0,\ n \in N$
$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n)$
$\leq \lambda(x_{n-1}, x_n) \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\}$
$\leq \lambda(x_{n-1}, x_n) \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\}$
$\leq \lambda(x_{n-1}, x_n) \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}$
Since
$\lambda(x_{n-1}, x_n) \leq \lambda\big(p(x_{n-1}, x_n)\big)$
So, we have
$$p(x_n, x_{n+1}) \leq \lambda(p(x_{n-1}, x_n)) \max\{p(x_{n-1}, x_n), , p(x_n, x_{n+1})\} \qquad (3.1)$$
There are two cases

(i) $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$
(ii) $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$
For case (i),From (3.1)
$$p(x_n, x_{n+1}) \leq \lambda\big(p(x_{n-1}, x_n)\big).\, p(x_n, x_{n+1})$$
Which is contradiction.
Now , for case (ii),from (3.1)
$$p(x_n, x_{n+1}) \leq \lambda\big(p(x_{n-1}, x_n)\big)\, p(x_{n-1}, x_n) \leq \cdots \leq \prod_{k=0}^{n-1} \lambda(p(x_k, x_{k+1}))p(x_0, Tx_0)$$
And
$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1})$$

$$\leq \lambda(x_n, x_{n-1}) \max\{p(x_n, x_{n-1}), p(x_n, Tx_n), p(x_{n-1}, Tx_{n-1})\}$$
$$\leq \ldots \leq \leq \prod_{k=0}^{n-1} \lambda(p(x_k, x_{k+1}))p(x_0, Tx_0)$$

It follows that

$$p(x_n, x_{n+1}) < p(x_0, Tx_0)$$

and

$$p(x_{n+1}, x_n) < p(Tx_0, x_0)$$

for all $n = 1, 2, \ldots$.

Now we prove that

$$p(x_0, x_n) \leq C$$

for some $C > 0$ and $n = 1, 2, 3, \ldots$

In fact,

$$p(x_1, x_{n+1}) = p(Tx_0, Tx_n)$$
$$\leq \lambda(x_0, x_n) \max\{p(x_0, x_n), p(x_0, Tx_0), p(x_n, Tx_n)\}$$
$$\leq \lambda(p(x_0, x_n))p(x_0, x_n)$$

and by the triangle inequality

$$p(x_0, x_n) \leq p(x_0, x_1) + p(x_1, x_{n+1}) + p(x_{n+1}, x_n)$$
$$p(x_0, x_n) \leq p(x_0, x_1) + \lambda(p(x_0, x_n))p(x_0, x_n) + p(Tx_0, x_0)$$

hence

$$p(x_0, x_n) < \frac{p(x_0, Tx_0) + p(Tx_0, x_0)}{1 - \lambda(p(x_0, Tx_n))}$$

Now if $p(x_0, Tx_n) \geq \alpha_0$ for a given $\alpha_0 > 0$, then by the monotonicity of $\lambda(p)$ it follows that

$$\lambda(p(x_0, Tx_n)) \leq (\alpha_0)$$

and therefore

$$p(x_0, x_n) < \frac{p(x_0, Tx_0) + p(Tx_0, x_0)}{1 - \lambda(\alpha_0)}$$
$$= C$$

On the other hand if $p(x_k, x_{k+1}) \geq \epsilon_0, k = 0, 1, \ldots, n-1$ for any $\epsilon_0 > 0$ then by monotonicity of λ it follows that

$$\lambda\, p(x_k, x_{k+1}) \geq \lambda(\epsilon_0)$$

and hence

$$p(x_n, x_{n+1}) < [\lambda(\epsilon_0)]^n\, p(Tx_0, x_0)$$

But $0 \leq \lambda(\epsilon_0) < 1$ by lemma 2.1 we have

$$\lim_{n \to \infty} p(x_n, x_{n+1}) = 0.$$

We shall show that $\{x_n\}$is a Cauchy sequence in (M, d). For $m > 0$,

$$p(x_n, x_{n+m}) \leq \prod_{k=0}^{n-1} \lambda(p(x_k, x_{k+m}))p(x_0, Tx_0)$$

If $p(x_k, x_{k+m}) \geq \epsilon_0$ for any given $\epsilon_0 > 0$ and $k = 0, 1, \ldots, n-1$ then

$$p(x_n, x_{n+m}) \leq [\lambda(\epsilon_0)]^n p(x_0, Tx_0) \to 0$$

as $n \to \infty$ and by lemma 2.1 we have that $\{x_n\}$ is a Cauchy sequence. Since (M,d) is complete, $\{x_n\}$ converges to some $z \in M$. Since $x_m \to z$ and $p(x_n, .)$ is lower semicontinuous,

$$p(x_n, z) \leq \lim_{n\to\infty} p(x_n, x_m) \leq \lambda^n(\epsilon_0)p(x_0, Tx_0)$$

so $\lim_{n\to\infty} p(x_n, z) = 0$.

On the other hand,

$$\begin{aligned} p(x_n, Tz) &= p(Tx_{n-1}, Tz) \\ &\leq \lambda(x_{n-1}, z) \max\{p(x_{n-1}, z), p(x_{n-1}, Tx_{n-1}), p(z, Tz)\} \\ &\leq \lambda(p(x_{n-1}, z))p(x_{n-1}, z) < p(x_{n-1}, z) \end{aligned}$$

so $\lim_{n\to\infty} p(x_n, Tz) = 0$.

and by lemma 2.2 we have $Tz = z$.

$Now,$

$$p(z,z) = p(Tz,Tz) \leq \lambda(z,z)\max\{p(z,z), p(z,Tz), p(z,Tz)\} < p(z,z)$$

$so\ p(z,z) = 0$.

If y = Ty then

$$\begin{aligned} p(z,y) &= p(Tz,Ty) \\ &\leq \lambda(z,y)\max\{p(z,y), p(z,Tz), p(y,Ty)\} < p(z,y) \end{aligned}$$

and

$$p(z,y) = 0$$

so by lemma 2.1 we have $z = y$.

Remarks:

$a.-$ In case $p = d$, (M,d) is a complete metric space and $T : M \to M$ is a Rakotch contraction then we get the Rakotch's theorem[10].

$b.-$ If (M,d)a complete metric space and $\lambda(x,y) = k,\ 0 \leq k < 1$ we get a generalization of the Banach Contraction Principle[8] and[15].

Theorem 3.2. Let (M,d) be a complete metric space, let p be a ω −distance on M and $T : M \to M$ is a mapping such that for some integer $m \in N$ T^m is an ω −Rakotch contraction. Then T has a unique fixed point, i.e., there exists $z \in M$ such that $Tz = z$ and moreover holds $p(z,z) = 0$.

Proof: Since for some $m \in N$ T^m is a ω −Rakotch contraction, then there exists a function $\lambda(x,y) \in \mathcal{F}$ such that

$$p(T^m x, T^m y) \leq \lambda(x,y)\max\{p(x,y), p(x,T^m x), p(y,T^m y)\}$$

for every $x, y \in M$. Hence by theorem 3.1 there exists a unique $z \in M$ such that $z = T^m z$ for $m \in N$ and $Tz = T(T^m z) = T^m(Tz)$ it follows that $z = Tz$.

Let us remark that in case $\lambda(x,y) = k, 0 \leq k < 1, p = d$ and (M,d) complete metric space we get the Chu-Diaz's Theorem[2].

Now we get another generalization of Rakotch's Theorem[10] using Maia's Theorem[11].

Theorem 3.3. Let M be a non empty set, d, and ρ two metrics on M, p and q their respective ω −distances on M and $T : M \to M$ a mapping. Suppose that:

$a.-$ $p(x,y) \leq q(x,y)$ for all $x, y \in M$.

$b.-$ (M,d) is a complete metric space.

$c. - T : (M, \rho) \to (M, \rho)$ is a ω −Rakotch contraction, *i.e.*, there exists $\lambda(x, y) \in \mathcal{F}$ such that

$$q(Tx, Ty) \leq \lambda(x, y) \max\{q(x, y), q(x, Tx), q(y, Ty)\}$$

for every $x, y \in M$.

Then there exists $z \in$ M such that $Tz = z$ and moreover $p(z, z) = 0$.

Proof: Let $x_0 \in M$ and define $x_n = T^n x_0, n \in N$. from (c), $\{x_n\}$ is a Cauchy sequence in (M, ρ). By (a) and lemma 2.2, $\{x_n\}$ is a Cauchy sequence in (M, d) and by (b) it converges. The rest of the proof is similar to Theorem 3.1.

Now we generalize a result given by Singh-Deb-Gardner in[13].

Theorem 3.4. Let $\epsilon \in (0, +\infty)$ be and let (M, d) be a complete ϵ-chainable metric space. If T is a mapping from M into itself satisfying,

$0 < d(x, y) < \epsilon$

implies

$$d(Tx, Ty) \leq \lambda(x, y)\, max\, \{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in M$ and $\lambda(x, y) \in F$. Then T has a unique $z \in M$ such that $z = Tz$.

Proof: Since (M, d) is ϵ-chainable for every $x, y \in M$ we define the function $p : M \times M \to [0, +\infty)$ as follows:

$$p(x, y) = inf\{\textstyle\sum_{i=1}^{n} d(x_{i-1}, x_i)\}$$

is an ϵ-chain joining x and y.

From lemma 2.2, p is a ω −distance on M satisfying $d(x, y) \leq p(x, y)$. Given $x, y \in M$ and any ϵ −chain $\{x_0, x_1, \ldots, x_n\}$ with $x_0 = x$ and $x_n = y$ we have for $i = 1, \ldots, n$,

$d(Tx_{i-1}, T\,x_i) \leq \lambda\big(d(x_{i-1}, x_i)\big) \max\{d(x_{i-1}, x_i), d(x_{i-1}, Tx_{i-1}), d(x_i, Tx_i)\} < \lambda(\epsilon)\epsilon < \epsilon$

Hence $Tx_0, \ldots, Tx_n$ is an ϵ-chain joining Tx and Ty, and

$$p(Tx, Ty) \leq \textstyle\sum_{i=1}^{n} d(Tx_{i-1}, Tx_i) \leq \sum_{i=1}^{n} \lambda\big(d(x_{i-1}, x_i) d(x_{i-1}, x_i)\big)$$

Since $\{x_0, x_1, \ldots, x_n\}$ is an arbitrary ϵ-chain we have

$$p(Tx, Ty) \leq \lambda(x, y) max\, \{p(x, y), p(x, Tx), p(y, Ty)\},$$

hence by theorem 3.1, T has a unique fixed point $z \in M, z = Tz$.

Remark: If $\lambda(x, y) = k,\ \ 0 \leq k < 1$ and $p = d$ we get the result due to Edelstein[4]. Finally, the following result generalizes Singh's theorem[12].

Theorem 3.5. Let $\epsilon \in (0, +\infty)$ be and let (M, d) a complete ϵ-chainable metric space. If T is a mapping from M into itself satisfying the condition,

$d(x, y) < \epsilon$ implies $d(T^m x, T^m y) \leq \lambda(x, y) d(x, y)$

for every $x, y \in M$, for $m \in M$ and $\lambda(x, y) \in \mathcal{F}$, then T has a unique fixed point in M.

Proof: As in theorem 3.4 we define p as follows:

$p(x, y) = inf\{\, d(x_{i-1}, x_i)\, / \{x_0, x_1, \ldots, x_n\}\ is\ an\ \epsilon - chain\ joining\ x\ and\ y\}$

By lemma 2.2, p is a ω −distance on M satisfying $d(x, y) \leq p(x, y)$. As in theorem 3.3 we have that T^m satisfies the condition

$$p(T^m x, T^m y) \leq \lambda(x, y) \max\{p(x, y), p(x, T^m x), p(y, T^m y)\}$$

for all $x, y \in M, m \in N$ and therefore by theorem 3.4 we conclude that T^m has a unique $z \in M$ such that $z = T^m z$. It follows that T has a unique fixed point z and moreover $p(z, z) = 0$. Finally, using the ideas of M.Telci-K.Tas[18] we obtain a generalization of Rakotch's theorem on noncomplete metric spaces.

Theorem 3.6. Let (M, d) be a noncomplete metric space and let p be a ω −distance on M. Let $T : M \to M$ be a ω −Rakotch contraction and suppose that there exists a point $u \in M$ such that

$$\theta(u) = inf\{\theta(x)/x \in M\}$$

where $\theta(x) = p(x, Tx)$ for all $x \in M$. Then u is a fixed point of T.

Proof: Suppose that $u \neq T(u)$, since otherwise u would be a fixed point of T. Now since T is a ω −Rakotch contraction there exists $\lambda(x, y) \in \mathcal{F}$ such that

$$p(Tx, Ty) \leq \lambda(x, y)\max\{p(x, y), p(x, Tx), p(y, Ty)\}$$

for all $x, y \in M$ and so

$$\theta(Tu) = p(Tu, T^2u) \leq$$
$$\lambda(u, Tu) \max\{p(u, Tu), p(u, Tu), p(Tu, T^2u)\}$$
$$\leq \lambda(p(u, Tu))p(u, Tu) < p(u, Tu) = \theta(u)$$

which is a contradiction.

REFERENCES

1. Caristi, J. "Fixed Point theorems for satisfying inwardness conditions", *Trans. A.M.S.* 215: 241–251 (1976).
2. Chu, S.C.; Diaz, J.B. "Remarks on a generalization of Banach's Principle of contraction mappings", *J. Math. Anal. Appl.* 11: 440–446 (1965).
3. Ciric, L.J. "A generalization of Banach's contraction principle", *Proc. A.M.S.* 45: 267–273 (1974).
4. Edelstein, M. "An extension of Banach's Contraction Principle", *Proc. A.M.S.* 12(7). 10a edici´on: 1995 (1961 & 1995).
5. Ekeland, I. "Non-convex minimization problems", *Bull. A.M.S.* 1: 443–474 (1979).
6. Kada, O.; Suzuki, T.; Takahashi, W. "Non convex minimization theorems and fixed point theorems in complete metric spaces", *Math. Japon.* 44: 381–391 (1996).
7. Kannan, R. "Some results on fixed points-II", *Amer. Math. Monthly* 76: 405– 408 (1969).
8. Morales, J.R. Generalizations of some fixed point theorems, (to appear).
9. Nadler Jr., S.B. "Multivalued contraction mappings", *Pacif. Journ. Math.* 30: 475–488 (1969).
10. Rakotch, E. "A note on contractive mappings", *Proc. A.M.S.* 13: 459–465 (1962).
11. Rus, I.A. "Seminar on fixed point theory", Preprint 3, Babes-Bolyai University, Faculty of Mathematics (1983).
12. Singh, S.P. "Fixed point theorems in metric spaces", *Riv. Mat. Univ. Parma* 3(1): 37–40 (1972).

13. Singh, K.L.; Deb, S.; Gardner, B. "On contraction mappings", *Riv. Mat. Univ. Parma* 3(2): 143–151 (1972).
14. Subrahmanyan, P.V. "Remarks on some fixed point theorems related to Banach's Contraction Principle", *J. Math. Phys. Sci.* 8: 445–457 (1974).
15. Suzuki, T.; Takahashi, W. "Fixed point theorems and characterizations of metric completeness", *Top. Meth. in Nonlinear Analysis* 8: 371–382 (1996).
16. Suzuki, T. "Several fixed point theorems in complete metric spaces", *Yokohama Mathematical Journ.* 44: 61–72 (1997).
17. Takahashi, W. "Existence theorems generalizing fixed point theorems for multivalued mappings", in *Fixed Point Theory and Applications*, Pitman Research Notes in mathematics, Series 252: 397–406 (1991).
18. Telci, M.; Tas, K. "Some fixed point theorems on an arbitrary metric space", *Math. Balk.* 6(3): 251–255 (1992).
19. Jose R. Morales "Generalization of Rakotch's fixed point theorem", *Revista De Matematica* 9(1): 25-33 (2002).