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Common Fixed Point Theorems in Fuzzy Metric Spaces Using (JCLR) Property

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ABSTRACT

The aim of this paper is to prove a common fixed point theorem via joint common limit in the range of mappings property (JCLR) in fuzzy metric space. We generalized the result of Cho *et al*.[7].

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Keywords: Common Fixed Points, Fuzzy Metric Space, Weak compatible map, Joint common limit in the range of mappings property (JCLR) property, Common limit range (CLRg) property, Implicit relation.

I. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh L.A.[20], in 1965, as a new way to represent the vagueness in every day life. In mathematical programming problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using fuzzy sets. It was developed extensively by many authors and used in various fields to use this concept in topology and analysis. Abbas M. *et al*.[1], Balasubramaniam *et al*.[3], Chauhan S. and Kumar S.[4], Chauhan S.[12], Kumar S., Fisher B.[13],SharmaS.[17] have defined fuzzy metric space in various ways. George and Veeramani[8] modified the concept of fuzzy metric space introduced by Kramosil and Michalek[11] in order to get the Hausdorff topology. Jungck[10] introduced the notion of compatible maps for a pair of self mapping.

Mishra *et al*.[14] extended the notion of compatible mappings (introduce by Junck[10] in metric space) under the name of asymptotically commuting maps and Singh B.[18] extended the notion of weakly compatible maps to fuzzy metric space. Grabiec M.[9], obtained the Banach contraction principle in fuzzy version. The study of common fixed point of non-compatible mappings is also of great interest due to Pant[15]. Popa V.[16] proved theorem for weakly compatible non-continuous mapping using implicit relation. In 2008 Altun I. *et al*.[2] proved common fixed point theorem on fuzzy metric space with an implicit relation. Sintunavarat *et al*.[19] introduced a new concept of property (CLRg). Chauhan et al.[5] utilize the notion of common limit range property to prove unified fixed point theorems for weakly compatible mapping in fuzzy metric spaces. Implicit relation and (CLRg) property are used as a tool for finding common fixed point of contraction maps. Recently Chouhan S.[6] utilizes the notion of JCLR to prove common fixed point theorems for weak compatible mappings in fuzzy metric spaces using (JCLR) property. We prove a common fixed point theorem for weakly compatible mapping with joint common limit in the range of mappings property (JCLR).

II. PRELIMINARIES

Definition 2.1 A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is called a continuous[18] t norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ $a,b,c,d\in[0,1]$. Example of t-norms are $a*b=ab$ and $a*b=\min\{a,b\}$.

Definition 2.2 A Triplet $(X,M,*)$ is called a fuzzy metric space[8] if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions: for all $x,y,z\in X$ and $t,s>0$,

[1] $M(x,y,t)>0$,

[2] $M(x,y,t)=M(y,x,t)$,

[3] $M(x,y,t)=1$ For all $t>0$ if and only if $x=y$,

[4] $M(x,y,.):[0,\infty)\rightarrow[0,1]$ is continuous,

[5] $M(x,y,t)*(y,z,s)\leq M(x,z,t+s)$

Note that $M(x,y,t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x=y$ with $M(x,y,t)=1$ for all $t>0$.

Definition 2.3 Let (X,d) be a metric space[8]. Define $a*b=\min\{a,b\}$ and

$$M(x,y,t)=\frac{t}{t+d(x,y)} \quad \text{for all} \quad x,y\in X$$

and all $t>0$.Then $(X,M,*)$ is a fuzzy metric space. It is called the fuzzy metric space induced b the metric d.

Definition 2.4 Let (X, M, $*$) be a fuzzy metric space[15]. Then

(a) A sequence $\{x_n\}$ in X is said to converges to x in X if for each $\in>0$ and each t > 0, there exists $n_0\in\mathbb{N}$ such that $M(x_n, x, t) > 1-\in$ for all $n\geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\in>0$ and each t > 0, there exists

$n_0 \in N$ such that M (x_n, x_m, t) > 1 −∈ for all $n, m \geq n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5 Let F and G be maps from a fuzzy metric space $(X, M, *)$ into itself. The maps[14] F and G are said to compatible (or asymptotically commuting) if

$$\lim_{n\to\infty} M(FGx_n, GFx_n, t) = 1$$

For all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = z \text{ for some } z \in X .$$

Definition 2.6 Two maps F and G, from a fuzzy metric space[18] (X, M,∗) into itself are said to be weak-compatible if they commute at their coincidence points,

i.e. $Fx = Gx$

$\Rightarrow FGx = GFx$

Remark 2.7 Let (F, G) be a pair of self-maps of a fuzzy metric space (X, M,∗).Then (F, G) is R-weakly commuting implies that (F, G) is compatible[18], which implies that (F, G) is weak compatible. But the converse is not true.

Definition 2.8 A pair (f,g) of self-mappings of a fuzzy metric space (X, M,∗) is said to satisfy the common limit range[5] of g property(briefly, (CLR_g) property),if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gv \text{ where } v \in X$$

Definition 2.9 Let $(X, M, *)$ be a fuzzy metric space and $P, Q, R, S : X \to X$ .The pair (P, Q) and (R, S) are satisfy the joint common limit in the range of mappings (JCLR) property[6] if there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Ry_n = \lim_{n\to\infty} Sy_n = Qv = Rv$$

for some $v \in X$ (A)

Remark 2.10 If $R = P, S = Q$ and $\{x_n\} = \{y_n\}$ in (A) then we get the definition of CLRg.

Definition 2.11 Let φ_6 denote the set of all continuous function[5] $\Phi : [0,1]^6 \to R$ satisfying the conditions:

[1] Φ is non-increasing in t_2, t_3, t_4, t_5 and t_6 .

[2] $\Phi(u, v, v, v, v, v) \geq 0$ Implies $u \geq v$ for all

$u, v \in [0,1]$

i.e. $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$

Lemma 2.12 Let $(X, M, *)$ be a fuzzy metric space[7]. If there exists $k \in (0,1)$ such that for all $x, y \in X, M(x, y, kt) \geq M(x, y, t)$, ∀ t > 0, then $x = y$.

Lemma 2.13 The only t-norm ∗ satisfying r ∗ r ≥ r for all $r \in [0,1]$ is the minimum[18] t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$.

III. MAIN RESULTS

Theorem 3.1 Suppose that P,Q,S and T be self maps of fuzzy metric space $(X,M,*)$ satisfying

3.1.1 (P,S) or (Q,T) satisfying the property (JCLR),

3.1.2 $\phi\begin{bmatrix} M(Px,Qy,\mathrm{kt}), M(Px,Sx,\mathrm{t}), M(Qy,\mathrm{Ty},\mathrm{t}), \\ M(Px,Ty,\mathrm{t}), M(Qy,Sx,\mathrm{t}), M(Sx,\mathrm{Ty},\mathrm{t}) \end{bmatrix} \geq 0$

3.1.3 $P(X) \subset T(X), Q(X) \subset S(X)$

3.1.4 One of $P(X), Q(X), S(X)$ and $T(X)$ is complete subspace of X .Then the point (P,S) and (Q,T) have a point of coincidence each. Moreover P,Q,S and T have common fixed point provided the pair (P,S) and (Q,T) commute pair wise. (i.e. $PS = SP, QT = TQ$).

Proof. Since the pair (P,S) and (Q,T) satisfies the property (JCLR), then there exists a sequence $\{\mathrm{x}_n\}$ and $\{\mathrm{y}_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Ty_n = Su = Tu$$

for some $u \in X$. Now we show that $Tu = Qu$.

We put $x = x_n$ and $y = u$ in (3.1.2)

$$\phi\begin{bmatrix} M(Px_n,Qu,\mathrm{kt}), M(Px_n,Sx_n,\mathrm{t}), M(Qu,\mathrm{Tu},\mathrm{t}), \\ M(Px_n,Tu,\mathrm{t}), M(Qu,Sx_n,\mathrm{t}), M(Sx_n,\mathrm{Tu},\mathrm{t}) \end{bmatrix} \geq 0$$

By using JCLR property

$$\phi\begin{bmatrix} M(Tu,Qu,\mathrm{kt}), M(Tu,Tu,\mathrm{t}), M(Tu,\mathrm{Qu},\mathrm{t}), \\ M(Tu,Tu,\mathrm{t}), M(Qu,Tu,\mathrm{t}), M(Tu,\mathrm{Tu},\mathrm{t}) \end{bmatrix} \geq 0$$

$$\phi\left[M(Tu,Qu,\mathrm{kt}),1,M(Tu,\mathrm{Qu},\mathrm{t}),1,M(Tu,Qu,\mathrm{t}),1\right] \geq 0$$

Since ϕ is non -increasing in second, fourth and fifth argument.

$$\phi\begin{bmatrix} M(Tu,Qu,\mathrm{kt}), M(Tu,\mathrm{Qu},\mathrm{t}), M(Tu,\mathrm{Qu},\mathrm{t}), M(Tu,\mathrm{Qu},\mathrm{t}), \\ M(Tu,Qu,\mathrm{t}), M(Tu,\mathrm{Qu},\mathrm{t}) \end{bmatrix} \geq 0$$

Therefore by the definition of implicit relation 2.11

$M(Tu,Qu,kt) \geq M(Tu,Qu,t)$

By using lemma 2.12 we have

$Tu = Qu$

We show that $Pu = Tu$.

We put $x = u$ and $y = y_n$ in (3.1.2).

$$\phi\begin{bmatrix} M(Pu,Qy_n,\mathrm{kt}), M(Pu,Su,\mathrm{t}), M(Qy_n,\mathrm{Ty}_n,\mathrm{t}), \\ M(Pu,Ty_n,\mathrm{t}), M(Qy_n,Su,\mathrm{t}), M(Su,\mathrm{Ty}_n,\mathrm{t}) \end{bmatrix} \geq 0$$

By using JCLR property

$$\phi\begin{bmatrix} M(Pu,Tu,\mathrm{kt}), M(Pu,Tu,\mathrm{t}), M(\mathrm{Tu},\mathrm{Tu},\mathrm{t}), \\ M(Pu,Tu,\mathrm{t}), M(Tu,\mathrm{Tu},\mathrm{t}), M(\mathrm{Tu},\mathrm{Tu},\mathrm{t}) \end{bmatrix} \geq 0$$

$$\phi\left[M(Pu,Tu,\mathrm{kt}), M(Pu,Tu,\mathrm{t}),1,M(Pu,Tu,\mathrm{t}),1,1\right] \geq 0$$

Since ϕ is non -increasing in third, fifth and sixth argument

$$\phi\begin{bmatrix} M(Pu,Tu,\mathrm{kt}), M(Pu,Tu,\mathrm{t}), M(Pu,Tu,\mathrm{t}), \\ M(Pu,Tu,\mathrm{t}), M(Pu,Tu,\mathrm{t}), M(Pu,Tu,\mathrm{t}) \end{bmatrix} \geq 0$$

Therefore by the definition of implicit relation 2.11

$M(Pu,Tu,kt) \geq M(Pu,Tu,t)$

By using lemma 2.12 we have

$Pu = Tu$

Now suppose that $v = Pu = Qu = Tu = Su$.

Since the pair (P,S) is weakly compatible $PSu = SPu \Rightarrow Pv = Sv$. Similarly the pair (Q,T) is also weak compatible then $QTu = TQu \Rightarrow Qv = Tv$.

Now we show that $Pv = v$.

We put $x = v$ and $y = u$ in (3.1.2)

$$\phi\begin{bmatrix} M(Pv,Qu,\mathrm{kt}), M(Pv,Sv,\mathrm{t}), M(Qu,\mathrm{Tu},\mathrm{t}), \\ M(Pv,Tu,\mathrm{t}), M(Qu,Sv,\mathrm{t}), M(Sv,\mathrm{Tu},\mathrm{t}) \end{bmatrix} \geq 0$$

By using JCLR property

$$\phi\begin{bmatrix} M(Pv,v,\text{kt}), M(Pv,Pv,\text{t}), M(v,v,\text{t}), \\ M(Pv,v,\text{t}), M(v,Pv,\text{t}), M(v,v,\text{t}) \end{bmatrix} \geq 0$$

$$\phi\left[M(Pv,v,\text{kt}),1,1,M(Pv,v,\text{t}),M(v,Pv,\text{t}),1\right] \geq 0$$

Since ϕ is non -increasing in second, third and sixth argument

$$\phi\begin{bmatrix} M(Pv,v,\text{kt}), M(Pv,v,\text{t}), M(Pv,v,\text{t}), \\ M(Pv,v,\text{t}), M(v,Pv,\text{t}), M(Pv,v,\text{t}) \end{bmatrix} \geq 0$$

Therefore by the definition of implicit relation 2.11

$M(Pv,v,kt) \geq M(Pv,v,t)$

By using lemma 2.12 we have

$Pv = v \Rightarrow v = Pv = Sv$

We Show that $Qv = v$.

We put $x = u$ and $y = v$ in (3.1.2)

$$\phi\begin{bmatrix} M(Pu,Qv,\text{kt}), M(Pu,Su,\text{t}), M(Qv,\text{Tv},\text{t}), \\ M(Pu,Tv,\text{t}), M(Qv,Su,\text{t}), M(Su,\text{Tv},\text{t}) \end{bmatrix} \geq 0$$

By using JCLR property

$$\phi\begin{bmatrix} M(v,Qv,\text{kt}), M(v,v,\text{t}), M(Qv,\text{Qv},\text{t}), \\ M(v,v,\text{t}), M(Qv,v,\text{t}), M(v,Qv,\text{t}) \end{bmatrix} \geq 0$$

$$\phi\left[M(v,Qv,\text{kt}),1,1,1,M(v,\text{Qv},\text{t}),M(\text{v},\text{Qv},\text{t})\right] \geq 0$$

Since ϕ is non -increasing in second, third and fourth argument

$$\phi\begin{bmatrix} M(v,Qv,\text{kt}), M(v,\text{Qv},\text{t}), M(v,\text{Qv},\text{t}), \\ M(v,\text{Qv},\text{t}), M(v,\text{Qv},\text{t}), M(\text{v},\text{Qv},\text{t}) \end{bmatrix} \geq 0$$

Therefore by the definition of implicit relation 2.11

$M(v,\text{Qv},kt) \geq M(v,\text{Qv},t)$

By using lemma 2.12 we have

$v = Qv$

Therefore we have $v = Pv = Qv = Sv = Tv$.

P,Q,S and T have common fixed point is a point v.

Uniqueness: Let z be another common fixed point of mapping P,Q,S and T.

We put $x = v$ and $y = w$ in (3.1.2)

$$\phi\begin{bmatrix} M(Pv,Qw,\text{kt}), M(Pv,Sv,\text{t}), M(Qw,\text{Tw},\text{t}), \\ M(Pv,Tw,\text{t}), M(Qw,Sv,\text{t}), M(Sv,\text{Tw},\text{t}) \end{bmatrix} \geq 0$$

By using JCLR property

$$\phi\begin{bmatrix} M(v,w,\text{kt}), M(v,v,\text{t}), M(w,w,\text{t}), \\ M(v,w,\text{t}), M(w,v,\text{t}), M(v,w,\text{t}) \end{bmatrix} \geq 0$$

$$\phi\left[M(v,w,\text{kt}),1,1,M(v,w,\text{t}),M(w,v,\text{t}),M(v,w,\text{t})\right] \geq 0$$

Since ϕ is non -increasing in second and third argument

$$\phi\left[M(v,w,\text{kt}),M(v,w,\text{t}),M(v,w,\text{t}),M(v,w,\text{t}),M(v,w,\text{t}),M(v,w,\text{t})\right] \geq 0$$

Therefore by the definition of implicit relation 2.11

$M(v,w,kt) \geq M(v,w,t)$

By using lemma 2.12 we have

$v = w$

Therefore P,Q,S and T have a unique common fixed point.

We can also prove the same result if the pair (P,S) satisfies the joint common limit in the range of mappings (JCLR) property. The prove is same when $T(X)$ is assumed to be complete subspace of X. The remaining two cases pertain essentially to previous cases. If let $A(X)$ is a complete subspace of X then $v \in P(X) \subset T(X)$ or $Q(X)$ is a complete subspace of X then $v \in Q(X) \subset S(X)$. Thus we can also establish that both the pair (P,S) and (Q,T) have a point of coincidence each.

Corollary 3.2 Let P and S be self maps of fuzzy metric space $(\text{X},\text{M},*)$ satisfying the following:

[3.2.1] (P, S) satisfies the joint common limit in the range of mappings (JCLR) property,

[3.2.2] $P(X) \subseteq S(X)$,

[3.2.3] $\phi\begin{bmatrix} M(Px, Py, kt), M(Px, Sx, t), M(Py, Sy, t), \\ M(Px, Sy, t), M(Py, Sx, t), M(Sx, Sy, t) \end{bmatrix} \geq 0$,

[3.2.4] One of $P(X)$ and $S(X)$ is a complete subspace of X.

Then the pairs (P, S) have a point of coincidence. Moreover P and S have a unique common fixed point provided the pairs (P, S) is weakly compatible.

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