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Packing Chromatic Number of Certain Graphs

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ABSTRACT

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k for which there exists a mapping $\pi : V(G) \to \{1,2,...,k\}$ such that any two vertices of color i are at distance at least $i+1$. It is a frequency assignment problem used in wireless networks, which is also called broadcast coloring. It is proved that packing coloring is NP-complete for general graphs and even for trees. In this paper, we give the packing chromatic number for splitting of bi star graph, sierpiński graph, broken wheel, jahangir graph and $P_4 \; K_q$.

Keywords: Packing chromatic number; Bistar graph; Sierpiński graph; Broken wheel; Jahangir graph.

1. INTRODUCTION

Let G be a connected graph and k be an integer, $k \geq 1$. A packing $k-$coloring of a graph is a mapping $\pi : V(G) \to \{1,2,...,k\}$ such that any two vertices of color i are at distance at least $i+1$. The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k for which G has packing $k-$coloring. The concept of packing coloring comes from the area of frequency assignment in wireless networks and was introduced by Goddard *et al.*[4] under the name broadcast coloring. The term packing chromatic number was introduced by Brešar[2]. In this paper we find the packing chromatic number for splitting of bistar graph, sierpiński graph, broken wheel, jahangir graph and $P_4 \; K_q$.

Proposition 1:[4] Let H be a subgraph of G. Then $\chi_\rho(H) \leq \chi_\rho(G)$.

Proposition 2:[4] Let $C_n, n \geq 5$ be a cycle. Then $\chi_\rho(C_n) = 4$ when $n \not\equiv 0 \ mod\ 4$.

2. MAIN RESULTS

Definition 1:[6]

The *splitting graph* of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G, i.e. $N(v) = N(v')$. The resultant graph is denoted by $S'(G)$.

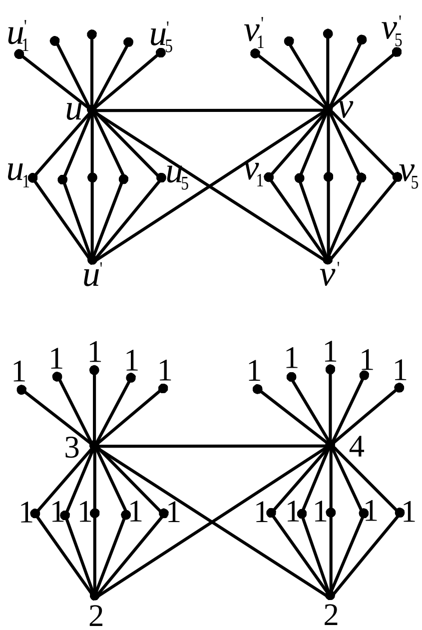

Figure 1: $\chi_\rho(S'(B_{5,5})) = 4$

Theorem 1:

Let $B_{n,n}, n \geq 3$ be a bistar graph.
Then $\chi_\rho(S'(B_{n,n})) = 4$

Proof: Color the vertices u_i, v_i, u_i', v_i' with 1, u', v' with 2 and u with 3 and v with 4. Thus $\chi_\rho(S'(B_{n,n})) \leq 4$. See Figure 1.

Since C_6 is a subgraph of $S'(B_{n,n})$, by Proposition 1 and 2, $\chi_\rho(S'(B_{n,n})) = 4$.

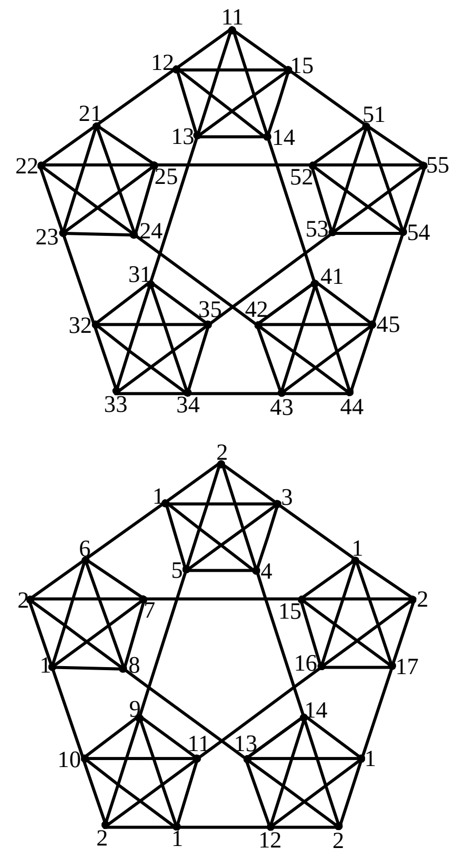

Figure 2: $\chi_\rho(S(2,5)) = 17$

Definition 2:[3]

The *Generalised Sierpiński Graph* $S(n,k),\ n \geq 1, k \geq 1$ is defined in the following way: $V(S(n,k)) = \{1, 2, ..., k\}^n$, two distinct vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ being adjacent if and only if there exists an $h \in \{1, 2, ..., n\}$ such that

(1) $u_t = v_t$, for $t = 1, ..., h-1$;

(2) $u_h \neq v_h$; and

(3) $u_t = v_h$ and $v_t = u_h$ for $t = h+1, ..., n$.

For convenience, we write the vertex $(u_1, u_2, ..., u_n)$ as $\langle u_1 u_2 ... u_n \rangle$. The vertices $\langle 1...1 \rangle, \langle 2...2 \rangle, ..., \langle k...k \rangle$ are called the *extreme vertices* of $S(n,k)$. For $k \geq 2$, $S(n,k)$ contains k copies of the graph $S(n-1,k)$ and k^{n-1} copies of the complete graph $S(1,k) = K_k$.

Theorem 2:

For $S(2,k),\ k \geq 3,$
$\chi_\rho(S(2,k)) = (k^2 - 2k) + 2$

Proof:

Since K_3 is a subgraph of $S(2,k)$, $\chi_\rho(S(2,k)) \geq 3$. The diameter of $S(2,k)$ is 3. Since $d(\langle ii \rangle, \langle jj \rangle) = 3,\ 1 \leq i, j \leq k; i \neq j$, at most k extreme vertices can be colored with 2. $S(2,k)$ contains k copies of K_k. Therefore, at most one vertex in each K_k receives color 1. Thus the number of vertices colored 1 and 2 must be at most $2k$. Further, no color greater than 2 can be used more than once. Thus remaining $k^2 - 2k$ vertices should receive distinct colors. Hence $\chi_\rho(S(2,k)) \geq (k^2 - 2k) + 2$. We give an algorithm to show that $S(2,k)$ using exactly $(k^2 - 2k) + 2$ colors.

PROCEDURE PACKING COLORING

Input: *Sierpiński Graph* $S(2,k),\ k \geq 3$

Algorithm:

Step 1: Give color 1 to $\langle i\,i+1 \rangle, 1 \leq i \leq k-1$ and $\langle k1 \rangle$.

Step 2: Give color 2 to $\langle ii \rangle, 1 \leq i \leq k$.

Step 3: Give distinct colors to the remaining vertices of $S(2,k)$ starting from 3.

Output: $\chi_\rho(S(2,k)) = (k^2 - 2k) + 2$

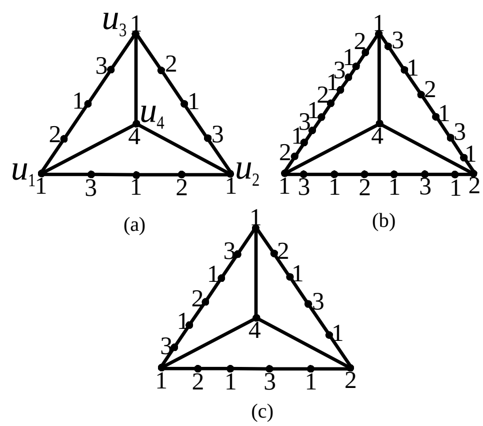

Figure 3: (a) $\chi_\rho(W(4,4,4)) = 4$
(b) $\chi_\rho(W(7,7,10)) = 4$ (c) $\chi_\rho(W(5,5,6)) = 4$

Proof of correctness:

Since $d(\langle i\,i+1 \rangle, \langle j\,j+1 \rangle) \geq 2$ and $d(\langle k1 \rangle, \langle i\,i+1 \rangle) \geq 2$ for $1 \leq i, j \leq k-1; i \neq j$, the vertices $\langle k1 \rangle$ and $\langle i\,i+1 \rangle$ are colored 1. Since $d(\langle ii \rangle, \langle jj \rangle) = 3$, for $1 \leq i, j \leq k; i \neq j$, the vertices $\langle ii \rangle$ are colored 2. Therefore $2k$ vertices are colored with 1 and 2. The total number of vertices in $S(2,k)$ is k^2. On giving distinct colors to remaining vertices starting from 3, $S(2,k)$ is colored with $(k^2-2k)+2$ colors. See Figure 2.

Definition 3: [5]

For integers a,b,c with $1 \le a \le b \le c$, the *broken wheel* $W(a,b,c)$ with three spokes is the graph constructed from a complete graph K_4 where $V(K_4)=\{u_1,u_2,u_3,u_4\}$, by inserting $(a-1)$ vertices $\{x_{1,1},x_{1,2},...,x_{1,a-1}\}$ along the edge (u_1,u_2), $(b-1)$ vertices $\{x_{2,1},x_{2,2},...,x_{2,b-1}\}$ along the edge (u_2,u_3), $(c-1)$ vertices $\{x_{3,1},x_{3,2},...,x_{3,c-1}\}$ along the edge (u_3,u_1).

Theorem 3:

For $W(a,b,c), c \ge b \ge a \ge 3$, $\chi_\rho(W(a,b,c))=4$, when

(1) $a=b=c \equiv 0 \; mod \; 4$

(2) $a+1=b+1 \equiv 0 \; mod \; 4$ and $c \not\equiv 0 \; mod \; 4$, m is even.

(3) $a-1=b-1 \equiv 0 \; mod \; 4$ and $c \not\equiv 0 \; mod \; 4$, m is even.

Proof:
Let the vertices of $W(a,b,c)$ be u_1,u_2,u_3,u_4. See Figure 3(a).

Case 1: $a=b=c \equiv 0 \; mod \; 4$
Fix the color 4 to u_4. Color the vertices of $W(a,b,c)$ in the outer cycle using the sequence 1312,... starting at u_1 in anticlockwise sense. Thus $\chi_\rho(W(a,b,c)) \le 4$. Since C_{a+2}, $a+2 \not\equiv 0 \; mod \; 4$ is a subgraph of $W(a,b,c)$ by Proposition 1 and 2 $\chi_\rho(W(a,b,c))=4$. See Figure 3(a).

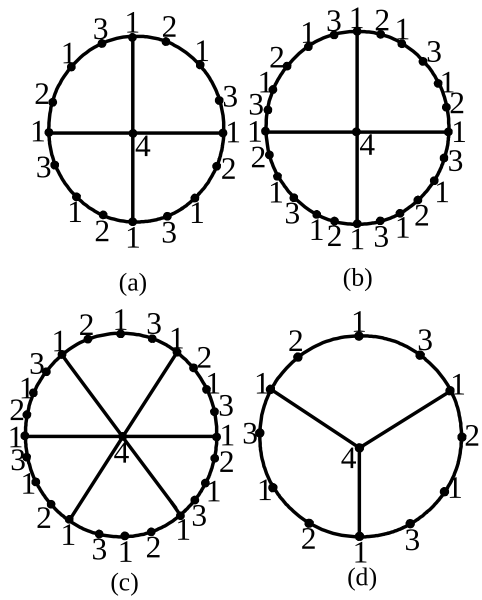

Figure 4: (a) $\chi_\rho(J_{4,4})=4$ (b) $\chi_\rho(J_{6,4})=4$ (c) $\chi_\rho(J_{4,6})=4$ (d) $\chi_\rho(J_{4,3})=4$

Case 2: $a+1=b+1 \equiv 0 \; mod \; 4$ and $c \not\equiv 0 \; mod \; 4$, m is even.
The proof is similar to Case 1. See Figure 3(b).

Case 3: $a-1=b-1 \equiv 0 \; mod \; 4$ and $c \not\equiv 0 \; mod \; 4$, m is even.

Fix the color 4 to u_4. Color the vertices of $W(a,b,c)$ in the outer cycle using the sequence 1213,... starting at u_1 in anticlockwise sense. Thus $\chi_\rho(W(a,b,c)) \le 4$. Since C_{a+2}, $a+2 \not\equiv 0 \; mod \; 4$ is a subgraph of $W(a,b,c)$ by Proposition 1 and 2 $\chi_\rho(W(a,b,c)) = 4$. See Figure 3(c).

Definition 4 [1]**:**

Jahangir graph $J_{s,m}, s \ge 2, m \ge 2$ is a graph on $sm+1$ vertices consisting of a cycle C_{sm} with on additional vertex which is adjacent to m vertices of C_{sm} at distance s to each other on C_{sm}.

Theorem 4:

For $J_{s,m}, s, m \ge 3$, $\chi_\rho(J_{s,m}) = 4$ when

(1) both s and m are even.

(2) $s \equiv 0 \; mod \; 4$ and m is odd

Proof:
Let the vertex which is connected to m vertices of C_{sm} in $J_{s,m}$ be u.

Case 1: both s and m are even.

Sub Case 1: $s \equiv 0 \; mod \; 4$ and $m \equiv 0 \; mod \; 4$
Fix color 4 to vertex u. Color the vertices of $J_{s,m}$ in the outer cycle using the sequence $1312, \ldots$ starting at any vertex of degree 3 in any direction. Thus $\chi_\rho(J_{s,m}) \le 4$. Since C_{s+2}, $s+2 \not\equiv 0 \; mod \; 4$ is a subgraph of $J_{s,m}$ by Proposition 1 and 2, $\chi_\rho(J_{s,m}) = 4$. See Figure 4(a).

Sub Case 2: $s \not\equiv 0 \; mod \; 4$ and $m \equiv 0 \; mod \; 4$
The proof is similar to Sub Case 1. Whereas C_{2s+2}, $2s+2 \not\equiv 0 \; mod \; 4$ is a subgraph of $J_{s,m}$ by Propostion 1 and 2 $\chi_\rho(J_{s,m}) = 4$. See Figure 4(b).

Sub Case 3: $s \equiv 0 \; mod \; 4$ and $m \not\equiv 0 \; mod \; 4$
The proof is similar to Sub Case 1. See Figure 4(c).

Case 2: $s \equiv 0 \; mod \; 4$ and m is odd
The proof is similar to Sub Case 1. See Figure 4(d).

Remark 1: $J_{s,3}$ is a special case of $W(a,b,c)$ when $a = b = c = s$.

Definition 5: The graph $G_{d,q}$ is defined as follows: $G_{d,q} = P_d \;\; K_q$. $G_{d,q}$ is a connected graph of diameter d.

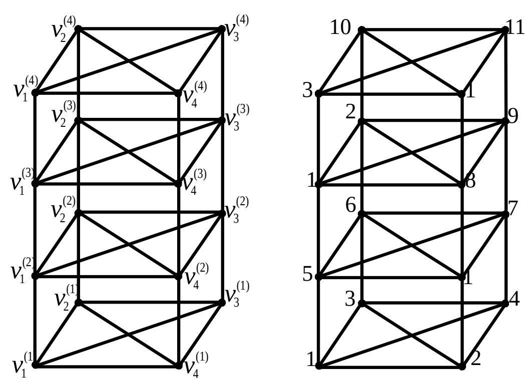

Figure 5: $\chi_\rho(G_{4,4}) = 11$

Theorem 5: For $G_{4,q}$, $q \ge 4$, $\chi_\rho(G_{4,q}) = (dq-8)+3$

Proof:

Since $q \geq 4$, $\chi_\rho(G_{4,q}) \geq 4$. Since $diam(G_{4,q}) = 4$ and $G_{4,q}$ contains 4 copies of K_q, at most two vertices with color 3 , two vertices with color 2 and four vertices with color 1 can be given. Thus the number of vertices colored 1,2 and 3 must be at most 8. Further, no color greater than 3 can be used more than once. Therefore remaining $(dq-8)$ vertices should receive distinct colors. Hence $\chi_\rho(G_{4,q}) \geq (dq-8)+3$.

We give an algorithm to show that $G_{4,q}$, $q \geq 4$ using exactly $(dq-8)+3$ colors.

Let $V(K_q) = \{v_1, ..., v_q\}$. Also $V(G_{4,q}) = \bigcup_{i=1}^{4} V^i$, $E(G_{4,q}) = \bigcup_{i=1}^{4} E^i \cup \bigcup_{j=1}^{4} E^j$, where $V^i = \{v_j^{(i)} : 1 \leq j \leq q\}$, $E^i = \{v_j^{(i)} v_k^{(i)} : 1 \leq j < k \leq q\}$, $E_j = \{v_j^{(i)} v_j^{(i+1)} : 1 \leq i \leq 3\}$.

PROCEDURE PACKING COLORING

Input: $G_{4,q} = P_d \quad K_q, q \geq 4$

Algorithm:

Step 1: Give color 1 to $v_1^{(1)}, v_4^{(2)}, v_1^{(3)}, v_4^{(4)}$.

Step 2: Give color 2 to $v_4^{(1)}, v_2^{(3)}$.

Step 3: Give color 3 to $v_2^{(1)}, v_1^{(4)}$.

Step 3: Give distinct colors to the remaining vertices of $G_{d,q}$ starting from 4.

Output: $\chi_\rho(G_{4,q}) = (dq-8)+3$

Proof of correctness: Since the distance between any two vertices $v_1^{(1)}, v_4^{(2)}, v_1^{(3)}, v_4^{(4)}$ is 2, the vertices $v_1^{(1)}, v_4^{(2)}, v_1^{(3)}, v_4^{(4)}$ are colored 1. Since $d(v_4^{(1)}, v_2^{(3)}) = 3$, the vertices $v_4^{(1)}$ and $v_2^{(3)}$ are colored 2. Since $d(v_2^{(1)}, v_1^{(4)}) = 4$, the vertices $v_2^{(1)}$ and $v_1^{(4)}$ are colored 3. Therefore 8 vertices are colored with 1,2 and 3 . The total number of vertices in $G_{4,q}$ is dq. On giving distinct colors to the remaining vertices starting from 4, $G_{4,q}$ is colored with $(dq-8)+3$ colors.

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