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A Mixed Quadrature Rule for Numerical Integration of Analytic Functions

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ABSTRACT

A mixed quadrature rule of blending Clenshaw-Curties five point rule and Gauss-Legendre 3 point rule is formed. The mixed rule has been tested and found to be more effective than that of its constituent Clenshaw-Curtis five point rule for the approximate evaluation of the integral of an analytic function over a line segment in complex plane. An asymptotic error estimate of the rule has been determined and the rule has been numerically verified.

Keywords: Quadrature rule, Asymptotic error, Analytic function, Numerical integration.

1. INTRODUCTION

There are several rules for the approximate evaluation of real integral

$$\int_{-1}^{1} f(x)\,dx \tag{1.1}$$

However there are only few quadrature rules for evaluating an integral type

$$F(f) = \int_{L} f(z)\,dx \tag{1.2}$$

Where L is directed line segment from the point $Z_0 - h$ to $Z_0 + h$ in the complex plane C and $f(z)$ is analytic in certain domain Ω containing the line segment L. Das and Pradhan[4] used Birkhoff-Young's quadrature rule to produce a mixed quadrature rule for analytic function.

In this light of Birkhoff and young's[2] interpolatory type of quadrature rule, we introduce the following Gauss-Legendrace 3 point quadrature rule

$$R_{Gl3}(f) = \frac{h}{9}\left[5f\left(z_0 - h\sqrt{\frac{3}{5}}\right) + 8f(z_0) + 5f\left(Z_0 + h\sqrt{\frac{3}{55}}\right)\right] \tag{1.3}$$

and this rule is of precision five using the transformation $Z = Z_0 + ht, t \in [-1,1]$ (due to lether[1]), we transformed the integral (1.2) to the form

$$h\int_{-1}^{1} f\left(Z_0 + ht\right) dt \qquad (1.4)$$

Add then made the approximation of this integral by applying standard quadrature rule meant for approximation evaluation of real definite integral (1.1). The rules so formed are termed as TRANSFORMED Rules for the numerical integration of (1.2). Das and Pradhan[3,4] have constructed quadrature rules combining rules of different type but of equal precision. Such rules are termed as MIXED QUADRATURE RULES.

Das R. B. and Jena S.[5], Das R.B. and Mohanty, S.[6,7] have constructed some mixed quadrature rules for analytic function.

In this paper we desired to construct a mixed quadrature rule of precision seven in the same vein for the approximation of the integral (1.2).

2. FORMULATION OF THE RULE

For the construction of the desired rule we choose the rule (1.3) and the Clenshaw Curtis five point rule.

$$R_{cc5}(f) = \frac{h}{15}\left[f(z_0+h) + f(z_0-h) + f\left(z_0+\frac{h}{\sqrt{2}}\right) + f\left(z_0-\frac{h}{\sqrt{2}}\right) + 12f(z_0)\right] \qquad (2.1)$$

Each of the rules (1.3) and (2.1) under considered is of precision five. Denoting the truncation errors by $E_{cc5}(f)$ & $E_{GL3}(f)$ in approximating the integral (1.2) by the rules (1.3) and (2.1) respectively, we have

$$I(f) = R_{cc5}(f) + E_{cc5}(f) \qquad (2.2)$$

$$\& \quad I(f) = R_{Gl3}(f) + E_{Gl3}(f) \qquad (2.3)$$

f is infinitely differentiable. Since it is assumed to be analytic in certain domain Ω containing the line segment L . So by using Tayler's expansion the truncation error associated with the quadratue rules under reference can be expressed as

$$E_{cc5}(f) = \frac{h^7 f^6(Z_0)}{315\times 5!} + \frac{h^9 f^8(Z_0)}{360\times 7!} + \cdots$$

$$E_{Gl3}(f) = \frac{4h^7 f^6(Z_0)}{525\times 5!} + \frac{11}{1125\times 7!}\times h^9 f^8(Z_0) + \cdots$$

Now multiplying the Equation (2.2) and (2.3) by $\frac{1}{5}$ & $\frac{-1}{12}$ respectively and then adding the resulting equations we obtain

$$I(f) = \frac{1}{7}\left[12R_{CC5}(f) - 5R_{Gl3}(f)\right] + \frac{1}{7}\left[12E_{CC5}(f) - 5E_{Gl3}(f)\right]$$

Or
$$I(f) = R_{CC5Gl3}(f) + E_{CC5Gl3}(f) \qquad (2.4)$$

Where

$$R_{cc5Gl3}(f) = \frac{1}{7}\left[12R_{cc5}(f) - 5R_{Gl3}(f)\right] \qquad (2.5)$$

This is desired quadrature rule of precision seven for the approximation evaluation of $I(f)$ & the truncation error generated in this approximation is given by

$$E_{cc5Gl3}(f)=\frac{1}{7}\left[12E_{cc5}(f)-5E_{Gl3}(f)\right] \quad (2.6)$$

The rule (2.5) may be called a MIXED TYPE rule as it is constructed from two different type of rules of the same precision (i.e. precision 5).

3. ERROR ANALYSIS

Let $f(Z)$ is analytic in the disc

$$\Omega_R=\{Z:|Z-Z_0|\le R>|h|\}$$

So that points

$Z_0, Z_0\pm h, Z_0\pm h\sqrt{\frac{3}{5}}, Z_0\pm\frac{h}{\sqrt{2}}$ are all interior to the disc Ω_R. Now using Taylor's expansion

$f(z)=\sum_{n=0}^{\infty}a_n(Z-Z_n)^n; a_n=\frac{1}{n!}f^n(z_0)$ in (2.6)

We obtain after simplification

$$E_{cc5Gl3}(f)=-h^9 f^8(Z_0)/(45\times 7!)... \quad (3.1)$$

From (3.1) we have the following

Theorem 3.1

If f is assumed to be analytic in a domain $\Omega\supset L$ then $E_{cc5Gl3}(f)=0(h^9)$. further it is shown by Lether that

$$|E_{Gl_3}(f)|\le|E_{CC5}(f)| \quad (3.2)$$

Now from (3.2) and the result given in (2.6). We have $|E_{cc5Gl3}(f)|\le|E_{Gl_3}(f)|$ thus $|E_{cc5Gl_3}(f)|\le|E_{Gl_3}(f)|\le|E_{cc5}(f)|$ i. e. the mixed quadrature rule (numerically integrated more accurately than its constituents Clenshaw- Curtis 5 point rule & Gauss legendre 3 point rule. This fact has been numerically verified in the next section.

4. NUMERICAL VERIFICATION

For the numerical verification of the facts of Theorem(3.1) are depicted in the following tables.

Table 1 Numerical Integration of $\int_{0.4}^{0.6}\frac{1}{z}dz$

Quadrature Rule	Approximation Value
$R_{cc5}(f)$	0.4054648
$R_{Gl3}(f)$	0.4054754
$S_{cc5Gl3}(f)$	0.4054649
Exact Value	0.4054651

Table 2 Numerical Integration of $\int_{-i/2}^{i/2}\cos z\,dz$

Quadrature Rule	Approximation Value
$R_{cc5}(f)$	1.0421904
$R_{Gl3}(f)$	1.0421901
$S_{cc5Gl3}(f)$	1.0421906
Exact Value	1.0421906

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