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Upper Limits to the Complex Growth Rate in Rivlin-Ericksen Fluid in the Presence of Rotation in a Porous Medium

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ABSTRACT

The thermal instability of a Rivlin-Ericksen viscoelastic fluid acted upon by uniform vertical rotation and heated from below in a porous medium is investigated. Following the linearized stability theory and normal mode analysis, the paper through mathematical analysis of the governing equations of Rivlin-Ericksen viscoelastic fluid convection with a uniform vertical rotation, for the case of rigid boundaries shows that the complex growth rate σ of oscillatory perturbations, neutral or unstable for all wave numbers, must lie inside the right half of the semi-circle

$$\sigma_r^{\ 2} + \sigma_i^{\ 2} \langle T_A \left(\frac{\varepsilon P_l}{P_l + \varepsilon F} \right)^2 ,$$

in a σ-plane, where T_A is the Taylor number, F is the viscoelasticity parameter, ε is the porosity, P_l is the medium permeability; which prescribes the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude in a rotatory Rivlin-Ericksen viscoelastic fluid heated from below. The result is important since it hold for all wave numbers and for rigid boundaries of infinite horizontal extension at the top and bottom of the fluid, and the exact solutions of the problem investigated in closed form, is not obtainable.

Keywords: Thermal convection, Rivlin-Ericksen Fluid, Rotation, PES, Rayleigh number, Taylor number.

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1. INTRODUCTION

Stability of a dynamical system is closest to real life, in the sense that realization of a dynamical system depends upon its stability. Right from the conceptualizations of turbulence, instability of fluid flows is being regarded at its root. The thermal instability of a fluid layer with maintained adverse temperature gradient by heating the underside plays an important role in Geophysics, interiors of the Earth, Oceanography and Atmospheric Physics, and has been investigated by several authors and a detailed account of the theoretical and experimental study of the onset of Bénard Convection in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar[1] in his celebrated monograph. The use of Boussinesq approximation has been made throughout, which states that the density changes are disregarded in all other terms in the equation of motion except the external force term. There is growing importance of non-Newtonian fluids in geophysical fluid dynamics, chemical technology and petroleum industry. Bhatia and Steiner[2] have considered the effect of uniform rotation on the thermal instability of a viscoelastic (Maxwell) fluid and found that rotation has a destabilizing influence in contrast to the stabilizing effect on Newtonian fluid. In another study Sharma[3] has studied the stability of a layer of an electrically conducting Oldroyd fluid[4] in the presence of magnetic field and has found that the magnetic field has a stabilizing influence. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's[4] constitutive relations. Two such classes of fluids are Rivlin-Ericksen's and Walter's (model B') fluids. Rivlin-Ericksen[5] has proposed a theoretical model for such one class of elastico-viscous fluids. Kumar *et al*.[6] considered effect of rotation and magnetic field on Rivlin-Ericksen elastico-viscous fluid and found that rotation has stabilizing effect; whereas magnetic field has both stabilizing and destabilizing effects. A layer of such fluid heated from below or under the action of magnetic field or rotation or both may find applications in geophysics, interior of the Earth, Oceanography, and the atmospheric physics. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable.

In all above studies, the medium has been considered to be non-porous with free boundaries only, in general. In recent years, the investigation of flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. When a fluid permeates a porous material, the gross effect is represented by the Darcy's law. As a result of this macroscopic law, the usual viscous term in the equation of Rivlin-Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu^{'}\frac{\partial}{\partial t}\right)q\right]$, where μ and $\mu^{'}$ are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and q is the Darcian (filter) velocity of the fluid. The problem of thermosolutal convection in fluids in a porous medium is of great importance in geophysics, soil sciences, ground water hydrology and astrophysics. Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of the comets, meteorites and interplanetary dust strongly suggest the importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. Thermal convection in porous medium is also of interest in geophysical system, electrochemistry and metallurgy. A comprehensive review of the literature concerning thermal convection in a fluid-saturated porous medium may be found in the book by Nield and Bejan[7]. Sharma *et al.*[8] studied the thermosolutal convection in Rivlin-Ericksen rotating fluid in porous medium in hydromagnetics with free boundaries only.

Pellow and Southwell[9] proved the validity of PES for the classical Rayleigh-Bénard convection problem. Banerjee *et al*.[10] gave a new scheme for combining the governing equations of thermohaline convection, which is shown to lead to the bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries and, Banerjee and Banerjee[11] established a criterion on characterization of non-oscillatory motions in hydrodynamics which was further extended by Gupta *et al*.[12]. However no such result existed for non-Newtonian fluid configurations in general and in particular, for Rivlin-Ericksen viscoelastic fluid configurations. Banyal[13] have characterized the oscillatory motions in couple-stress fluid.

Keeping in mind the importance of non-Newtonian fluids, the present paper is an attempt to prescribe the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude, in a layer of incompressible Rivlin-Ericksen viscoelastic fluid heated from below in a porous medium in the presence of uniform vertical rotation opposite to force field of gravity, when the bounding surfaces are of infinite horizontal extension, at the top and bottom of the fluid are rigid.

2. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we Consider an infinite, horizontal, incompressible Rivlin-Ericksen viscoelastic fluid layer, of thickness d, heated from below so that, the temperature and density at the bottom surface z = 0 are T_0 and ρ_0, and at the upper surface z = d are T_d and ρ_d respectively, and that a uniform adverse temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ is maintained. The gravity field $\vec{g}(0,0,-g)$ and uniform vertical rotation $\vec{\Omega}(0,0,\Omega)$ pervade on the system. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.

Let p, ρ, T, , α, g and $\vec{q}(u,v,w)$ denote respectively the fluid pressure, fluid density temperature, thermal coefficient of expansion, gravitational acceleration and filter velocity of the fluid. Then the momentum balance, mass balance, and energy balance equation of Rivlin-Ericksen fluid through porous medium, governing the flow of Rivlin-Ericksen fluid in the presence of uniform vertical vertical rotation (Rivlin and Ericksen[5]; Chandrasekhar[1] and Sharma *et al.*[6]) are given by

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}\left(\vec{q}.\nabla\right)\vec{q}\right]=-\left(\frac{1}{\rho_0}\right)\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right), \qquad (1)$$

$$\nabla.\vec{q}=0, \qquad (2)$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T, \qquad (3)$$

Where $\frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon^{-1}\vec{q}.\nabla$,

stand for the convective derivatives. (4)

Here $E=\varepsilon+(1-\varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_i}\right)$, (5)

is a constant and while ρ_s, c_s and ρ_0, c_i, stands for the density and heat capacity of the solid (porous matrix) material and the fluid, respectively, ε is the medium porosity and $\vec{r}(x,y,z)$.

The equation of state is

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right], \qquad (6)$$

Where the suffix zero refer to the values at the reference level z = 0. In writing the equation (1), we made use of the Boussinesq approximation, which states that the density variations are ignored in all terms in the equation of motion except the external force term. The kinematic viscosity ν, kinematic viscoelasticity ν', thermal diffusivity κ, and the coefficient of thermal expansion α are all assumed to be constants.

The steady state solution is

$$\vec{q}=(0,0,0),\ \rho=\rho_0(1+\alpha\beta z),\ T=-\beta z+T_0, \qquad (7)$$

Here we use the linearized stability theory and the normal mode analysis method. Consider a small perturbations on the steady state solution, and let $\delta\rho$, δp, θ and $\vec{q}(u,v,w)$ denote respectively the perturbations in density ρ, pressure p, temperature T and velocity $\vec{q}(0,0,0)$. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature is given by

$$\delta\rho=-\rho_0(\alpha\theta). \qquad (8)$$

Then the linearized perturbation equations of the Rinlin-Ericksen fluid reduces to

$$\frac{1}{\varepsilon}\frac{\partial \vec{q}}{\partial t}=-\frac{1}{\rho_0}(\nabla\delta p)-\vec{g}(\alpha\theta)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right), \qquad (9)$$

$$\nabla.\vec{q}=0, \qquad (10)$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta\,, \tag{11}$$

3. NORMAL MODE ANALYSIS

Analyzing the disturbances into two-dimensional waves, and considering disturbances characterized by a particular wave number, we assume that the Perturbation quantities are of the form

$$[w,\theta,\varsigma]=[W(z),\Theta(z),Z(z)]\exp\left(ik_x x+ik_y y+nt\right), \tag{12}$$

Where k_x, k_y are the wave numbers along the x- and y-directions, respectively, $k=\left(k_x^{\,2}+k_y^{\,2}\right)^{\frac{1}{2}}$, is the resultant wave number, n is the growth rate which is, in general, a complex constant; $\varsigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ denote the z-component of vorticity, $W(z),\Theta(z)$ and Z(z) are the functions of z only.

Using (12), equations (9)-(11), within the framework of Boussinesq approximations, in the non-dimensional form transform to

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\left(D^2-a^2\right)W=-Ra^2\Theta-T_A DZ\,, \tag{13}$$

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]Z=DW\,, \tag{14}$$

And

$$\left(D^2-a^2-Ep_1\sigma\right)\Theta=-W\,, \tag{15}$$

Where we have introduced new coordinates (x',y',z') = (x/d, y/d, z/d) in new units of length d and $D=d/dz'$. For convenience, the dashes are dropped hereafter. Also we have substituted $a=kd,\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$ is the thermal Prandtl number; $P_l=\frac{k_1}{d^2}$ is the dimensionless medium permeability, $F=\frac{\nu'}{d^2}$ is the dimensionless viscoelasticity parameter of the Rivlin-Ericksen fluid; $R=\frac{g\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number; and $T_A=\frac{4\Omega^2 d^4}{\nu^2\varepsilon^2}$ is the Taylor number. Also we have Substituted $W=W_\oplus$, $\Theta=\frac{\beta d^2}{\kappa}\Theta_\oplus$, $Z=\frac{2\Omega d}{\nu\varepsilon}Z_\oplus$ and $D_\oplus=dD$, and dropped $(\oplus)$ for convenience.

We now consider the case where both the boundaries are rigid and are maintained at constant temperature and then the perturbations in the temperature is zero at the boundaries. The appropriate boundary conditions with respect to which equations (13)-(15), must possess a solution are

W = DW = 0, $\Theta=0$ and Z=0, at z = 0 and z = 1. (16)

Equations (13)--(15), along with boundary conditions (16), pose an eigenvalue problem for σ and we wish to characterize σ_i, when $\sigma_r\geq 0$.

4. MATHEMATICAL ANALYSIS

We prove the following lemma:

Lemma 1: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 |Z|^2 dz \langle \frac{1}{|\sigma|^2 \left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)^2} \int_0^1 |DW|^2 dz \ .$$

Proof: Further, multiplying equation (14) with its complex conjugate, and integrating by parts each term on both sides of the resulting equation for an appropriate number of times and making use of appropriate boundary conditions (16), we get

$$\left[|\sigma|^2\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)^2+\frac{1}{P_l^2}+\frac{2\sigma_r}{P_l}\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)\right]\int_0^1 |Z|^2 dz = \int_0^1 |DW|^2 dz \ , \tag{17}$$

Now F $\rangle$ 0 and $\sigma_r \geq 0$, therefore the equation (17), give

$$\int_0^1 |Z|^2 dz \langle \frac{1}{|\sigma|^2 \left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)^2} \int_0^1 |DW|^2 dz \ , \tag{18}$$

This completes the proof of lemma**.**
We prove the following theorems:

Theorem 1: If R$\rangle$ 0, F$\rangle$0, $T_A\rangle 0$, $P_l\rangle 0$, $p_1\rangle 0$, $\sigma_r \geq 0$ and $\sigma_i \neq 0$ then the necessary condition for the existence of non-trivial solution (W,Θ,Z) of equations (13) – (15), together with boundary conditions (16) is that

$$|\sigma|^2 \langle T_A \left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2 .$$

Proof: Multiplying equation (13) by W^* (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1 W^*\left(D^2-a^2\right)W dz = -Ra^2\int_0^1 W^*\Theta\, dz - T_A\int_0^1 W^* DZdz \ , \tag{19}$$

Taking complex conjugate on both sides of equation (15), we get

$$\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* = -W^* , \tag{20}$$

Therefore, using (20), we get

$$\int_0^1 W^*\Theta dz = -\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz , \tag{21}$$

Also taking complex conjugate on both sides of equation (14), we get

$$\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]Z^* = DW^* , \tag{22}$$

Therefore, using (22) and making use of appropriate boundary conditions (16), we get

$$\int_0^1 W^* DZdz = -\int_0^1 DW^* Zdz = -\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^* Zdz \ , \tag{23}$$

Substituting (21) and (23), in the right hand side of equation (19), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1 W^*\left(D^2-a^2\right)W dz = Ra^2\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz + T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^* Zdz , \tag{24}$$

Integrating the terms on both sides of equation (24) for an appropriate number of times and making use of the appropriate boundary conditions (16), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz$$
$$=Ra^2\int_0^1\left(\left|D\Theta\right|^2+a^2\left|\Theta\right|^2+Ep_1\sigma^*\left|\Theta\right|^2\right)dz$$
$$-T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^*F)\right]\int_0^1\left|Z\right|^2dz, \quad (25)$$

Now equating the imaginary parts on both sides of equation (25), and cancelling $\sigma_i(\neq 0)$ throughout, we get

$$\left[\frac{1}{\varepsilon}+\frac{F}{P_l}\right]\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz$$
$$=\left[-Ra^2Ep_1\int_0^1\left|\Theta\right|^2dz+T_A\left\{\frac{1}{\varepsilon}+\frac{F}{P_l}\right\}\int_0^1\left|Z\right|^2dz\right], \quad (26)$$

Now R $\rangle$ 0, $\varepsilon \rangle 0$ and $T_A\rangle$ 0, utilizing the inequalities (18), the equation (26) gives,

$$\left[\frac{1}{\varepsilon}+\frac{F}{P_l}\right]\left[1-\frac{T_A}{\left|\sigma\right|^2}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2\right]\int_0^1\left|DW\right|^2dz+I_1\langle 0, \quad (27)$$

Where

$$I_1=\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)a^2\int_0^1\left|W\right|^2dz+Ra^2Ep_1\int_0^1\left|\Theta\right|^2dz,$$

and therefore , we must have

$$\left|\sigma\right|^2\langle T_A\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2 \quad (28)$$

Hence, if $\sigma_r \geq 0$ And $\sigma_i \neq 0$, then

$$\left|\sigma\right|^2\langle T_A\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2. \quad (29)$$

And this completes the proof of the theorem.

Theorem 2: For stationary convection the Rivlin-Ericksen viscoelastic fluid behaves like an ordinary Newtonian fluid i. e. for $T_A=0$ implies that $\sigma_r=0$ and $\sigma_i=0$, when both the bounding surfaces are rigid.

Proof: The inequality (32), can be written as

$$\sigma_r^{\ 2}+\sigma_i^{\ 2}\langle T_A\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2,$$

If $T_A=0$, then we necessarily have, $\sigma_r=0$ And $\sigma_i=0$,

Thus for stationary convection the Rivlin-Ericksen viscoelastic fluid in a porous medium behaves like an ordinary Newtonian fluid, when both the bounding surfaces are rigid and it mathematically establishes the result of Kumar *et al*.[6].

This completes the proof.

5. CONCLUSIONS

The inequality (29) for $\sigma_r \geq 0$ and $\sigma_i \neq 0$, can be written as

$$\sigma_r^{\ 2}+\sigma_i^{\ 2}\langle T_A\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2,$$

The essential content of the theorem, from the point of view of linear stability theory is that for the configuration of Rivlin-Ericksen viscoelastic fluid in a porous medium of infinite horizontal extension heated form below, having top and bottom bounding surfaces rigid, in the presence of uniform vertical rotation parallel to the force field of gravity, the complex growth rate of an arbitrary oscillatory motions of growing amplitude, must lie inside a semi-circle in the right half of the $\sigma_r \sigma_i$ - plane whose centre is at the origin and radius is $\sqrt{T_A}\left(\frac{\varepsilon P_l}{P_l + \varepsilon F}\right)$, where T_A is the Taylor number, F is the viscoelasticity parameter, ε is the porosity and P_l is the medium permeability.

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