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New Families of Square Sum Graphs

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ABSTRACT

In this paper Square Sum labeling of new families of graphs using graph operations on cycles with parallel chords are shown with illustrations. We prove that cycles with parallel chords are square sum graphs and extended the result to show that graphs obtained by attaching an arbitrary number of pendant edges at a vertex of degree 2 of cycle with parallel chords, duplication of a vertex of degree 2 of cycle with parallel chords and crown with parallel chords are square sum graphs. We also prove that chain of even cycles with parallel chords, swing with parallel chords are square sum graphs.

AMS Subject Classification: 05C78.

Keywords: Square sum labeling, Cycles with parallel chords, Chain of even cycles, Swing of cycles with parallel chords.

1. INTRODUCTION

Rosa[13] introduced β-valuations in 1967 which was the origin of graph labeling and it was renamed as graceful labeling by Golomb[7]. In graph theory, labeling of a graph is an assignment of integers to the vertices or edges under some constraints. Several labelings have been introduced by researchers for the past five decades. For a detailed survey on graph labeling we refer Gallian[4]. Here a finite simple and undirected graph $G = (V, E)$ where $|V(G)|=p$ and $|E(G)|= q$ without loops or multiple edges is considered. Ajitha, Arumugam and Germina[1] introduced square sum labeling and several graphs that are proved to be square sum are seen in[1,2,5,6,12,15,17,19,22,23]. Some of the graphs that are proved to admit odd even sum labeling, sequential labeling and square difference labeling are in[9,11,18]. Various labelings on cycle related graphs, cycles with zig zag chords, cycles with parallel chords are discussed in[3,8,16,20,21]. In this paper the main focus is on new families of square sum graphs obtained by graph operations on cycles with parallel chords. We give below some definitions which are relevant to this paper.

Definition 1.1[1]

Let G be a graph with p vertices and q edges. A bijective mapping $f: V(G) \rightarrow \{0,1,2,...p-1\}$ is said to be square sum labeling if the induced function $f^*: E(G) \rightarrow I$ (Set of Positive integers) defined by $f^*(xy) = [f(x)]^2 + [f(y)]^2$ is injective for every edge xy. A graph which satisfies square sum labeling is called a square sum graph.

Definition 1.2[16]

C_n denotes a cycle on n vertices and by joining two otherwise non-adjacent vertices of a cycle by an edge, a chord of a cycle is obtained. A cycle with parallel chords is defined as a graph G obtained from a cycle C_n ($n \geq 6$) with consecutive vertices $v_0, v_1, \ldots v_{n-1}, v_0$ by adding the chords $v_1 v_{n-1}$, $v_2 v_{n-2}$, $\ldots v_\alpha v_\beta$, where $\alpha = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 2$ if n is odd and $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 1$ if n is even. Then G has n vertices and M edges where $M = (3n-3)/2$ if n is odd and $M = (3n-2)/2$ if n is even.

Definition 1.3[14]

Consider n copies of C_{2m}. Chain of Cycles is defined as the graph obtained by identifying $v_{i,m}$ with $v_{i+1,m}$ for $i = 1,2,\ldots n-1$ where $v_{i,1}$, $v_{i,2}$ $\ldots v_{i,2m}$ are the vertices of i^{th} copy of C_{2m}.

Definition 1.4[10]

Swing is a graph obtained by considering s copies of C_n and joining a vertex from each copy of C_n to a common vertex c' which is denoted by S^n_s.

2. MAIN RESULTS

Theorem 2.1: Every cycle C_n ($n \geq 6$) with parallel chords is a square sum graph.

Proof: The cycle C_n with parallel chords having vertices v_0, v_1, $\ldots$ v_{n-1} is denoted by G. As seen in definition 1.2 G has n vertices and M edges. The vertex labeling $f: V(G) \rightarrow \{0,1,2,\ldots.n-1\}$ is defined as follows for the two cases that are considered here depending on n.

$f(v_i) = 2i - 1$, $1 \leq i \leq n/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd

$f(v_{n-i}) = 2i$, $0 \leq i \leq n/2 - 1$ if n is even and $0 \leq i \leq (n-1)/2$ if n is odd

From the above it is seen that f is bijective. The edge set $E(G) = E_1 \text{U} E_2 \text{U} E_3 \text{U} E_4$ where

$E_1 = \{v_i v_{i+1}, 1 \leq i \leq n/2 - 1$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd$\}$

$E_2 = \{v_{n-i}\ v_{n-(i+1)}, 0 \leq i \leq n/2 - 2$ if n is even and $0 \leq i \leq (n-3)/2$ if n is odd$\}$

$E_3 = \{v_i v_{n-i}, 1 \leq i \leq n/2 - 1$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd$\}$

$E_4 = \{v_0\ v_1, v_{n/2}\ v_{n/2+1}$ if n is even and $v_0\ v_1$ if n is odd$\}$.

The induced function $f^*: E(G) \rightarrow I$ (Set of Positive integers) is defined as follows.

$f^*(v_i v_{i+1}) = 8i^2 + 2$, $1 \leq i \leq n/2 - 1$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd

$f^*(v_{n-i}\, v_{n-(i+1)}) = 8i^2 + 8i + 4$, $0 \le i \le n/2 - 2$ if n is even and $0 \le i \le (n-3)/2$ if n is odd
$f^*(v_i\, v_{n-i}) = 8i^2 - 4i + 1$, $1 \le i \le n/2 - 1$ if n is even and $1 \le i \le (n-1)/2$ if n is odd
$f^*(v_0\, v_1) = 1$
$f^*(v_{n/2}\, v_{n/2+1}) = 2n^2 - 6n + 5$ if n is even

It is evident from the above labeling that the edge labels of E_1, E_2 are even and that of E_3 is odd. To show that the edge labels of E_1, E_2 are distinct we assume on the contrary that they are same.

For the edges of E_1 and E_2 :
if $i \ne j$, $1 \le i \le n/2 - 1$ and $0 \le j \le n/2 - 2$, let
$f^*(v_i\, v_{i+1}) = f^*(v_{n-j}\, v_{n-(j+1)})$ Then
$8i^2 + 2 = 8j^2 + 8j + 4$
$4(i^2 - j^2) = 4j + 1$

Here the left hand side is an even integer whereas the right hand side is an odd integer, a contradiction. Hence the edge labels of E_1 and E_2 are distinct. It is seen that all the edges of G have distinct labels and G is a square sum graph. An illustration is given in Fig. 1a and Fig. 1b

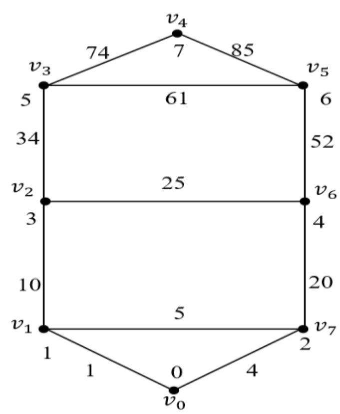

Fig. 1a: Square Sum labeling of C_8 with parallel chords

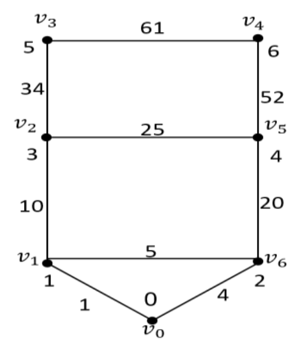

Fig. 1b: Square Sum labeling of C_7 with parallel chords

As an extension of the above theorem we have proved the following results given as corollaries

Corollary 1: The graph obtained by attaching an arbitrary number of pendant edges at a vertex of degree 2 of cycle C_n ($n \ge 6$) with parallel chords is a square sum graph.

Proof: Consider G as cycle C_n with parallel chords. The n vertices of cycle are $v_0, v_1, \ldots v_{n-1}$. Let G' be the graph obtained by attaching 'g' pendant edges at a vertex $v_{\frac{n}{2}}$ of cycle C_n of G. The pendant vertices are $d_1, d_2, \ldots d_g$. Then G' has $n+g$ vertices and $M+g$ edges where $M = \frac{3n-2}{2}$ if n is even and $M = \frac{3n-3}{2}$ if n is odd. The vertex labeling $f : V(G') \to \{0,1,2,\ldots n+g-1\}$is defined as follows:

$f(d_i) = n + i - 1$, $1 \le i \le g$.

The labeling for the vertices $v_0, v_1, \ldots v_{n-1}$ is same as in theorem 2.1
The edge set is given by $E(G') = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where E_1, E_2, E_3, E_4 are as in theorem 2.1 and $E_5 = \{v_{\frac{n}{2}} d_1,\ v_{\frac{n}{2}} d_2,\ \ldots .\ v_{\frac{n}{2}} d_g\}$. The induced edge labeling for edge set E_5 is given by
$f^*(v_{\frac{n}{2}} d_i) = (n-1)^2 + (n + i - 1)^2$, $1 \le i \le g$ if n is even and
$f^*(v_{\lceil \frac{n}{2} \rceil} d_i) = (n-1)^2 + (n + i - 1)^2$, $1 \le i \le g$ if n is odd.
For the edges of E_1, E_2, E_3, E_4 the labeling remains the same as in Theorem 2.1. Hence the graph under consideration is a square sum graph.

Corollary 2: Duplication of a vertex of degree 2 of cycle C_n $(n \ge 6)$ with parallel chords is a square sum graph

Proof: Cycle C_n with parallel chords is denoted by G. Let $v_0, v_1, \ldots v_{n-1}$ be the n vertices of G. Let G' be the graph obtained by duplication of a vertex of degree 2 say v_0 of G and let the new vertex be v_0' so that $N(v_0) = N(v_0')$. Using definition 1.2 G' has n+1 vertices and M +2 edges.
The vertex labelling $f: V(G') \rightarrow \{0,1,2,\ldots n\}$ is defined as follows.
$f(v_0') = n$.
The labeling for the vertices $v_0, v_1, \ldots v_{n-1}$ is same as in Theorem 2.1.
The edge set $E(G') = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where $E_5 = \{v_0' v_1, v_0' v_{n-1}\}$ and E_1, E_2, E_3, E_4 are same as in theorem 2.1. The induced function $f^*: E(G') \rightarrow I$ is defined as follows.
$f^*(v_0' v_1) = n^2 + 1$
$f^*(v_0' v_{n-1}) = n^2 + 4$.
The labeling remains the same as in Theorem 2.1 for the edges of E_1, E_2, E_3, E_4. All the edges receive distinct labels and G' is a square sum graph.

Corollary 3: Every Crown $C_n \odot K_1$ $(n \ge 6)$ with parallel chords is a square sum graph.

Proof: A cycle C_n with parallel chords is considered. By attaching a pendant edge at each vertex of cycle C_n with parallel chords, a Crown $C_n \odot K_1$ with parallel chords is obtained. Let this graph be G'. The vertices of cycle C_n are denoted by $v_0, v_1, \ldots v_{n-1}$ and let $v_0', v_1', \ldots v_{n-1}'$ be the pendant vertices. Then G' has $2n$ vertices and $M + n$ edges.
A bijective function $f: V(G') \rightarrow \{0,1,2,\ldots .2n-1\}$ is defined as follows.
$f(v_i') = 2i + n - 1$, $1 \le i \le n/2$ if n is even and $1 \le i \le (n-1)/2$ if n is odd,
$f(v_{n-i}') = i + n$, $0 \le i \le n/2 - 1$ if n is even and $0 \le i \le (n-1)/2$ if n is odd.
The labeling for the vertices $v_0, v_1, \ldots ., v_{n-1}$ is same as in theorem 2.1.
$E(G') = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ be the edge set where E_1, E_2, E_3, E_4 are same as in Theorem 2.1 and $E_5 = \{v_i v_i', 0 \le i \le n-1\}$.
The induced function $f^*: E(G') \rightarrow I$ is defined as follows.
$f^*(v_i' v_i) = n^2 + 8i^2 + 4i(n-2) - 2(n-1)$, $1 \le i \le n/2$ if n is even and $1 \le i \le (n-1)/2$ if n is odd,
$f^*(v_{n-i}' v_{n-i}) = n^2 + 8i^2 + 4in$, $0 \le i \le n/2 - 1$ if n is even and $0 \le i \le (n-1)/2$ if n is odd.
The labeling remains the same for all the edges of E_1, E_2, E_3, E_4 as in theorem 2.1. G' is a square sum graph.

Theorem 2.2: Chain of even cycles $C_{n,s}$ $(n \geq 6)$ with parallel chords is a square sum graph.

Proof: Consider s copies of even cycle C_n with parallel chords. For j=1,2,3,.….s let $v_{0,j}$, $v_{1,j}$,….. $v_{n-1,j}$ be the n vertices of j^{th} copy of cycle C_n with parallel chords. Chain of cycles $C_{n,s}$ with parallel chords is the graph obtained from s copies of even cycle C_n with parallel chords by identifying $v_{n/2,j}$ with $v_{0,j+1}$ for $j = 1,2,....s-1$. Let this graph be denoted by G. Then G has $s(n-1) + 1$ vertices and sM edges where $M = (3n - 2) / 2$.
Define the bijective function $f : V(G) \longrightarrow \{0,1,2,\ldots s(n-1)\}$ as follows.
For $j = 1,2,\ldots s$
$f(v_{i,j}) = 2i - 1 + (n-1)(j-1)$, $1 \leq i \leq n/2$
$f(v_{n-i,j}) = 2i + (n-1)(j-1)$, $0 \leq i \leq (n-2)/2$
From the above it is evident that the labels of all the vertices are distinct.
Edge set of G is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where
$E_1 = \{v_{i,j} v_{i+1,j}$ $1 \leq i \leq n/2 - 1$ and $1 \leq j \leq s\}$
$E_2 = \{v_{n-i,j} v_{n-(i+1),j}$ $0 \leq i \leq n/2 - 2$ and $1 \leq j \leq s\}$
$E_3 = \{v_{i,j} v_{n-i,j}$ $1 \leq i \leq n/2 - 1$ and $1 \leq j \leq s\}$
$E_4 = \{v_{0,j} v_{1,j}, v_{n/2,j} v_{n/2+1,j}$ and $1 \leq j \leq s\}$
The induced function $f^* : E(G) \longrightarrow I$ (Set of Positive integers) is defined as follows.
For $j = 1,2,\ldots s$
$f^*(v_{i,j} v_{i+1,j}) = (2i + j(n-1) - n)^2 + (2i + j(n-1) - n + 2)^2$, $1 \leq i \leq n/2 - 1$
$f^*(v_{n-1,j} v_{n-(i+1),j}) = (2i + j(n-1) - n + 1)^2 + (2i + j(n-1) - n + 3)^2$, $0 \leq i \leq n/2 - 2$
$f^*(v_{i,j} v_{n-i,j}) = (2i + j(n-1) - n)^2 + (2i + j(n-1) - n + 1)^2$, $1 \leq i \leq n/2 - 1$
$f^*(v_{0,j} v_{1,j}) = ((j-1)(n-1))^2 + ((j-1)(n-1) + 1)^2$
$f^*(v_{n/2,j} v_{n/2+1,j}) = (j(n-1) - 1)^2 + (j(n-1))^2$

From theorem 2.1 it is seen that if $j = 1$, the edge labels of the first copy are distinct. As the corresponding vertex labels of the successive copies are in an increasing sequence from the vertex labeling defined above, the edge labels are also in an increasing sequence for the successive copies and are distinct. Hence G is a square sum graph. An illustration is given in Fig. 2

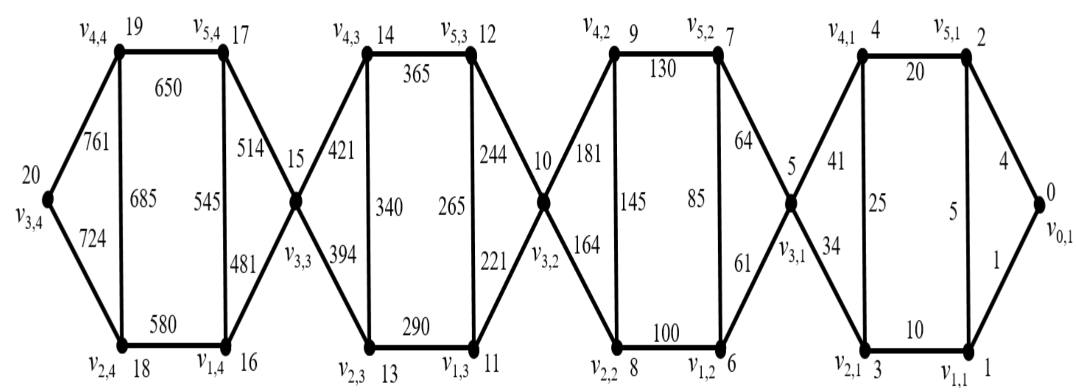

Fig.2: Square Sum labeling of $C_{6,4}$ with Parallel Chords

Theorem 2.3*:* Every Swing $S^n{}_s$ with parallel chords is a square sum graph

Proof: s copies of cycle C_n ($n \geq 6$) with parallel chords is considered. For j=1,2,3,…s let $v_{0,j}$, $v_{1,j}$, … $v_{n-1,j}$ be the n vertices of j^{th} copy of cycle C_n with parallel chords. Swing with parallel chords is obtained from s copies of cycle C_n with parallel chords by joining a vertex from each copy of C_n namely $v_{0,j}$ to a common vertex c'. Let this graph be denoted by $S^n{}_s$ with parallel chords which has sn +1 vertices and $sM + s$ edges where $M = (3n–2) / 2$ if n is even and $M = (3n -3) / 2$ if n is odd. Define the vertex labeling $f : V(S^n{}_s) \longrightarrow \{0,1,2,\ldots sn\}$as follows:

$f(c') = 0$
For $j = 1,2,3,\ldots s$
$f(v_{0,j}) = j$
$f(v_{i,j}) = s(2i – 1) + (2j-1)$, $1 \leq i \leq n/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd
$f(v_{n-i,j}) = 2j + s(2i-1)$, $1 \leq i \leq n/2 – 1$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd
The edge set $E(S^n{}_s) = E_1 U E_2 U E_3 U E_4 U E_5$ is given below.

For $j = 1,2,\ldots.s$
$E_1 = \{v_{i,j}\, v_{i+1,j}\ 1 \leq i \leq n/2 – 1$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd$\}$
$E_2 = \{v_{n-i,j}\, v_{n-(i+1),j}\ 1 \leq i \leq n/2 – 2$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd$\}$
$E_3 = \{v_{i,j}\, v_{n-i,j}\ 1 \leq i \leq n/2 - 1$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd$\}$
$E_4 = \{v_{0,j}\, v_{1,j}, v_{0,j}\, v_{n-1,j}\ v_{n/2,j}\, v_{n/2+1,j}$ if n is even and $v_{0,j}\, v_{1,j}, v_{0,j}\, v_{n-1,j}$ if n is odd$\}$
$E_5 = \{v_{c'}\, v_{0,j}\}$

Define the induced function $f^* : E(S^n{}_s) \longrightarrow I$ (Set of Positive integers) as follows.
For $j = 1,2,\ldots.s$
$f^*(v_{i,j}\, v_{i+1,j}) = [(2j – 1) + s(2i-1)]^2 + [(2j – 1) + s(2i + 1)]^2$, $1 \leq i \leq n/2 – 1$ if n is even &
$1 \leq i \leq (n- 3)/2$ if n is odd
$f^*(v_{n-i,j}\, v_{n-(i+1),j}) = [2j + s(2i -1)]^2 + [2j + s(2i + 1)]^2$, $1 \leq i \leq n/2 - 2$ if n is even &
$1 \leq i \leq (n-3)/2$ if n is odd
$f^*(v_{i,j}\, v_{n-i,j}) = [(2j – 1) + s(2i-1)]^2 + [2j + s(2i - 1)]^2$, $1 \leq i \leq n/2 - 1$ if n is even &
$1 \leq i \leq (n-1)/2$ if n is odd
$f^*(v_{0,j}\, v_{1,j}) = j^2 + [2+ j(s-1)]^2$,
$f^*(v_{0,j}\, v_{n-1,j}) = j^2 + [3+ j(s-1)]^2$
$f^*(v_{n/2,j}\, v_{n/2+1,j}) = [s(n-1) + j]^2 + [s(n-3) + 2j]^2$ if n is even
$f^*(v_{c'}\, v_{0,j}) = j^2$

As the corresponding vertex labels of the successive copies are in an increasing sequence, the edge labels defined above are also in an increasing sequence for the corresponding edges in successive copies and are distinct. Hence $S^n{}_s$ is a square sum graph. An illustration is given in Fig. 3

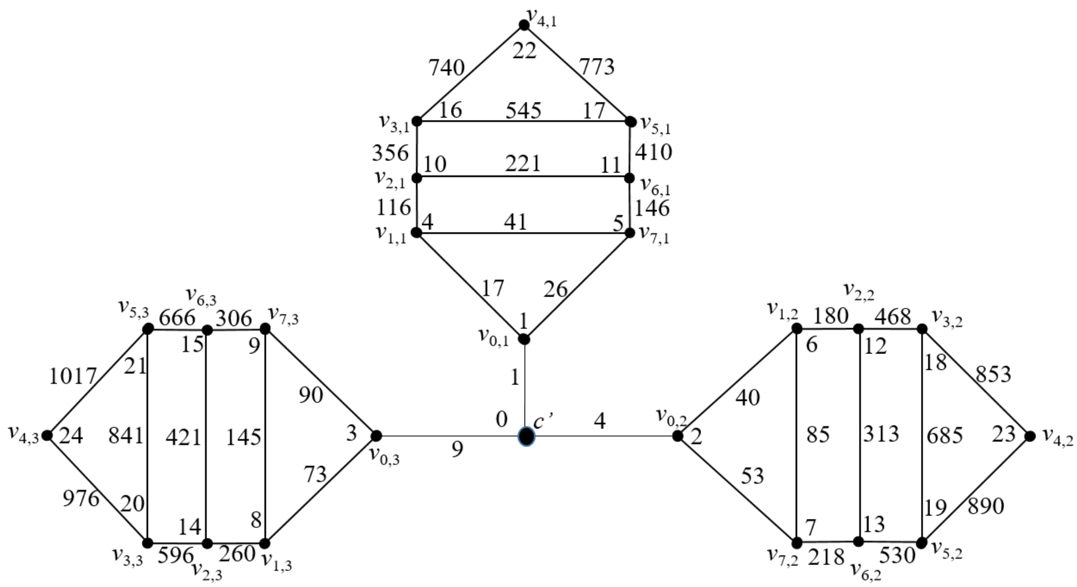

Fig.3: Square Sum labelling of Swing $S^8 4$ with Parallel Chords

CONCLUSION

In this paper we have obtained new families of graphs that are Square Sum graphs using graph operations on cycles with parallel chords.

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