Journal of Computer and Mathematical Sciences, *Vol.8(12), 788-801 December 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Star–in–coloring of Some Special Graphs

A. Sugumaran and P. Kasirajan

Department of Mathematics,
Government Arts College, Tiruvannamalai – 606 603, Tamil Nadu, INDIA.
email: sugumaranaruna@gmail.com, kasirajanmaths@gmail.com.

(Received on: December 18, 2017)

ABSTRACT

A proper coloring of a graph $G = (V, E)$ is a mapping $f: V \to N$such that if $v_i v_j \epsilon\ E$ then $f(v_i) \neq f(v_j)$. In this paper we investigate the lower and upper bounds for the star–in–chromatic number of the graphs such as cycle, regular cyclic, gear, fan, double fan, web and complete binary tree. In addition we have given the general coloring pattern of all these graphs and their star–in–chromatic number.

AMS Subject Classification: 05C15, 05C20.

Keywords: Coloring, Star–in–coloring, Star–in–chromatic number.

1. INTRODUCTION

Coloring of graphs is an active research area in graph theory. Many variations of chromatic number of graphs have been studied in literature. The concept of acyclic coloring of graphs was introduced by Grunbaum[7], also he introduced the concept of star–coloring of graphs. A graph G is called star–coloring if no path of length three is bicolored. The star–coloring of graphs have been investigated by Fertin, *et al*.[5] and Nesetril, *et al*.[9]. A digraph G is said to be in–coloring if any path of length two with end vertices of same color are always directed towards the middle vertex. Motivated through the concepts of star–coloring and in–coloring, Sudha and Kanniga[10] have introduced the star–in–coloring graphs.

For notation and graph theory terminology we refer to Harary[6]. Let $G = (V, E)$be a simple, connected digraph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E, each element of E is a directed edge. An orientation of a graph G is obtained by applying an orientation for each edge $e = v_i v_j \epsilon\ E$ either from v_i to v_j or $v_j$tov_i. We provide the basic definitions of graph theory which are necessary for our present investigations.

Definition 1.1 A proper *coloring* of a graph G is a mapping $f: V \to \{1,2,3, \dots\}$ such that if $e = v_i v_j \epsilon\ E$, then $f(v_i) \neq f(v_j)$.

Definition 1.2 A graph G is said to be *k–colorable* if we can assign one of k–colors to each vertex so that adjacent vertices have different colors.

If G is k–colorable, but not (k-1) colorable, then we say that the *chromatic number* of graph G denoted by $\chi\ (G)$ is k.

Definition 1.3 A *star–coloring* of a graph G is a proper coloring of the graph with the condition that no path of length three (P_4) is 2–colored.

An *n–star–coloring* of a graph G is a star–coloring of G using at most n colors.

Definition 1.4 An *in–coloring* of a graph G is a proper coloring of the graph G if there exist any path P_3 of length two with the end vertices are of the same color, then the edges of P_3 are oriented towards the central vertex.

Definition 1.5 A graph G is said to admit *star–in–coloring orientation* if it satisfies the following conditions.

1. No path of length three (P_4) is bicolored.
2. If any path of length two (P_3) with end vertices of same color, then the edges of P_3 are directed towards the middle vertex.

Definition 1.6 The minimum number of colors required for the star–in–coloring of a graph G is called the *star–in–chromatic number* of G and is denoted by $\chi_{si}(G)$.

First we describe the star–in–coloringof a simple graph as shown in Figure 1.1.

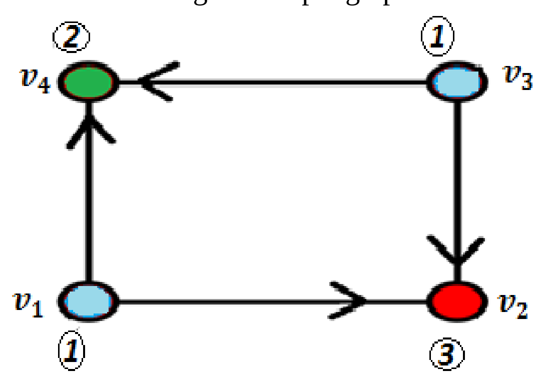


Figure 1.1: Star–in–coloring of Cycle C_4.

The vertices v_1 and v_3 are assigned with the color 1, the vertex v_4 is assigned with the color 2 and the vertex v_2 is assigned with the color 3.

In this graph we see that no two adjacent vertices have the same color, no path on four vertices is bicolored, each and every edge in a path of length two in which end vertices have same color are oriented towards the central vertex. Hence it is star–in–colored with orientation. Further the star–in–chromatic number of the above graph is 3.

Definition 1.7A graph $G = (V, E)$ in which $|V| > 3$ and maximum number of chords are drawn without form a triangle and the graph is regular, then the graph is called *a regular cyclic* graph. We denotea regular cyclic graph with p vertices and degree of each vertex $n$is denoted by RC(p, n).

Definition 1.8 A connected acyclic graph is called a *tree*. A binary tree is a tree in which only one vertex of degree two and each of the remaining vertices is of degree one or three. A vertex of degree two in a binary tree is called its root vertex. In a binary tree a vertex v is said to be at level l if v is at a distance l from the root vertex.

Definition 1.9 A binary tree with level n is said to be *complete* if each level l of the binary tree contains exactly 2^l vertices, where $0 \leq l \leq n$. A *complete binary tree* with level n is denoted by BT_n.
Note that the complete binary tree BT_ncontains$|V| = 1 + 2 + 2^2 + \cdots + 2^n$vertices and $|E| = |V| - 1$ edges.

2. MAIN RESULTS

In this section, we have to find the lower and upper bounds of star–in–chromatic number of some of the standard graphs. First we find the upper and lower bounds of star–in–chromatic number of cycle C_n.

Theorem 2.1
The cycle C_n admits star–in–coloring and its star–in–chromaticnumber satisfies the inequality $3 \leq \chi_{si}(C_n) \leq 4$, where n is even.

Proof: Consider a cycle C_n with the vertex set $V = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ and edge set $E = \{v_i\, v_{i+1} : 0 \leq i \leq n\text{-}1,\ v_n = v_0\}$. Then $|V| = n$ and $|E| = n$.
We define $f: V \to \{1, 2, 3, \dots\}$ as follows. We consider the following two cases:

Case1: $For\ n \equiv 0\ (mod\ 4)$
For each $i = 1, 2, 3, \dots, n$, we assign

$$f(v_i) = \begin{cases} 1, if\ i \equiv 1\ (mod\ 2) \\ 2, if\ i \equiv 2\ (mod\ 4) \\ 3, if\ i \equiv 0\ (mod\ 4) \end{cases}$$

In this case,$\chi_{si}(C_n) = 3$.

Case 2: $For\ n \equiv 2\ (mod\ 4)$
For each $i = 1, 2, 3, \dots, n$, we assign $f(v_0) = 4$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

In this case,$\chi_{si}(C_n) = 4$.

From the above cases, we conclude that $3 \leq \chi_{si}(C_n) \leq 4$.

Remark: The star–in–coloring of C_n is not possible, when n is odd. Since if we assign alternate colors for consecutive vertices, we obtain the initial and final vertices are of the same colors. On the other hand, if we assign a new color to the final vertex, then the in–coloring property is not satisfied.

Illustration 2.1.1 The star–in–coloring of cycle C_8 is shown in Figure 2.1.1.

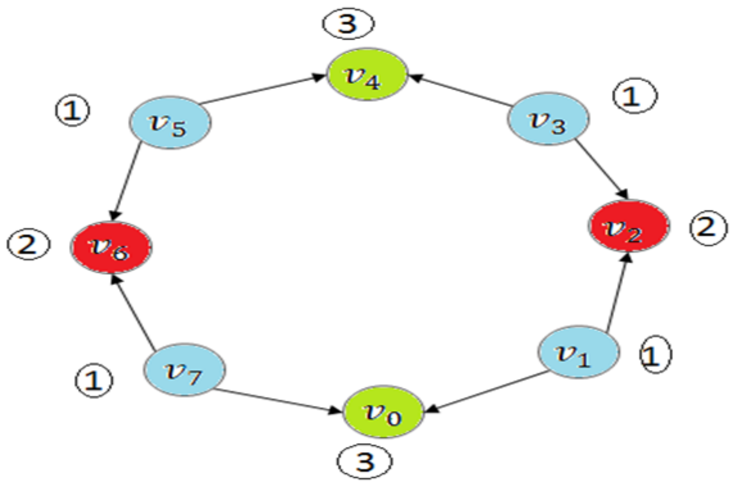
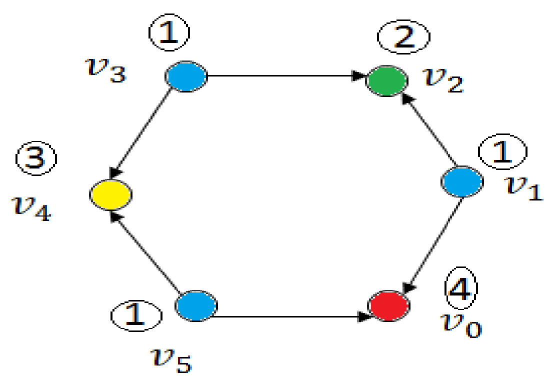


Figure 2.1.1: Cycle c_8

In cycle C_8 we assign color 1 to the vertices v_1,v_3, $v_5$and v_7, assign the color 2 tothe vertices v_2 and v_6 and color 3 is assigned to the remaining vertices.

The star–in–chromatic number of C_8 *is* $\chi_{si}(C_8) = 3$.

Illustration 2.1.2 The star–in–coloring of cycle c_6 is shown in Figure 2.1.2.



The vertices v_1,v_3, and$v_5$are assigned the color 1,whereas the vertices v_2, v_4 and v_0 are assigned with colors 2, 3 and 4 respectively.
The star–in–chromatic number of C_6 is $\chi_{si}(C_6) = 4$.

STAR–IN–COLORING OF REGULAR CYCLIC GRAPH

Theorem 2.2

The regular cyclic graph $\mathrm{RC}(p,n)$ admits star–in–coloring and its star–in–chromaticnumber is $n+1$, where p is an even integer and $p > 3$.

Proof: Consider a regular cyclic graph $\mathrm{RC}(p,n)$ which consists of $2n$ vertices and n^2 edges, where n is the degree of the each vertex. Let the vertices be denoted by $v_1, v_2, \dots, v_{2n}$.

Define a function $f: V \to \{1,2,3,\dots\}$ as follows.

We assign

$$f(v_i) = \begin{cases} 1, & if\ \ i \equiv 1\ (mod\ 2) \\ \frac{i}{2}+1, & if\ \ i \equiv 2\ (mod\ 2) \end{cases}$$

By using the above pattern of coloring the regular cyclic graph is star–in–colored.

The star–in–chromatic number of $\mathrm{RC}(p,n)$ is $\chi_{si}(\mathrm{RC}(p,n)) = n+1$.

Illustration 2.2.1 Consider a regular cyclic graph $\mathrm{RC}(10,5)$. As per the definition, which consists of 10 vertices and 25 edges.
The vertices $v_1, v_3, v_5, v_7$and v_9 are assigned the color 1. The vertices v_2, v_4, v_6, v_8 and v_{10} are assigned with colors 2,3,4,5 and 6 respectively.

The star–in–chromatic number of $\mathrm{RC}(10,5)$ is $\chi_{si}\big(\mathrm{RC}(10,5)\big) = 6$.

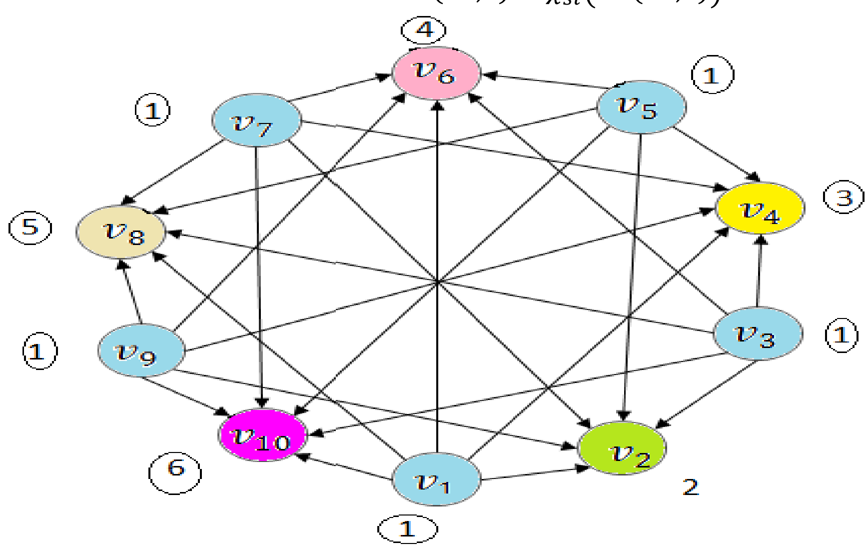


Figure 2.2.1: Regular cyclic graph RC(10, 5)

Remark: The graph RC(p,n) is not star–in–coloring, when p is odd, since at least one of the edges is left without orientation.

STAR–IN–COLORING OF GEAR GRAPH

Theorem 2.3

The gear graph G_n for all $n \geq 3$ admits star–in–coloring and its star–in–chromatic number satisfies the inquality $4 \leq \chi_{si}(G_n) \leq 5$.

Proof: Consider a gear graph G_n which consists of $2n-1$ vertcies and $3(n-1)$ edges. The central vertex is denoted by v_0 and the other vertices are denoted by $v_1, v_2, \dots, v_{2n-2}$.

We define $f: V \to \{1,2,3,\dots\}$ as follows. we consider the following two cases:

Case 1: For n is odd.

We assign $f(v_0) = 4$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

In this case $\chi_{si}(G_n) = 4$.

Case 2: For n is even.

We assign $f(v_0) = 5$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1 (mod 2) \\ 2, & if\ i \equiv 2 (mod 4) and\ i < 2n-2 \\ 3, & if\ i \equiv 0 (mod 4) and\ i > 0 \\ 4, & if\ i = 2n-2. \end{cases}$$

The above pattern of coloring of the graph G_n satisfies the star–in–coloring conditions.
In this case $\chi_{si}(G_n) = 5$. Hence the star–in–chromatic number of G_n satisfies $4 \leq \chi_{si}(G_n) \leq 5$.

Illustration 2.3.1 Consider a gear graph G_5. It consists of 9 vertices and 12 edges.

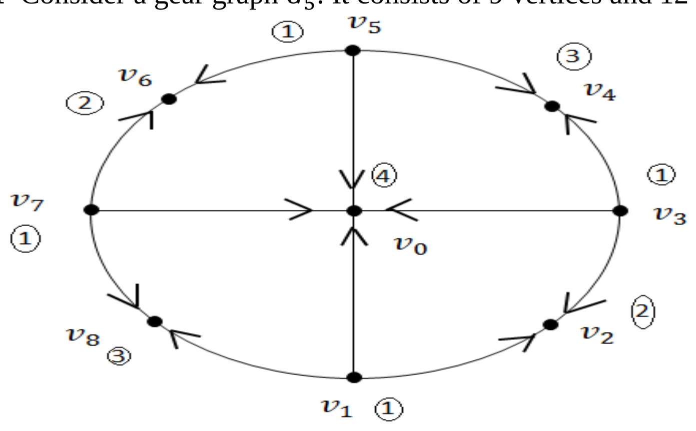


Figure 2.3.1: Gear graph G_5

The vertices v_1, v_3, v_5 and v_7 are assigned the color 1. The vertices v_2 and v_6 are assigned the color 2. The vertices v_4 and v_8 are assigned the color 3. The vertex v_0 is assigned the color 4.

The star–in–chromatic number of G_5 is $\chi_{si}(G_5) = 4$.

Illustration 2.3.2 Consider a gear graph G_4. It consists of 7 vertices and 9 edges.
The vertices v_1, v_3 and v_5 are assigned the color 1. The vertices v_2, v_4, v_6 and $v_0$are assigned the colors 2,3,4 and 5 respectively.

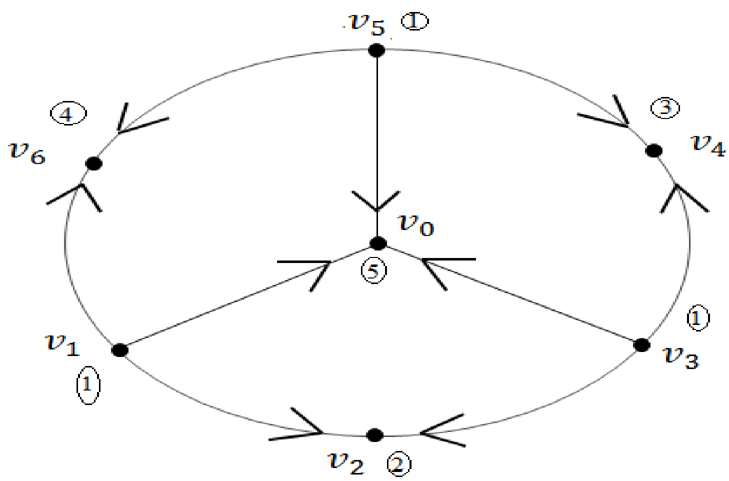


Figure 2.3.2: Gear graph G_4

The star–in–chromatic number of G_4 is $\chi_{si}(G_4) = 5$.

STAR–IN–COLORING OF FAN GRAPH

Theorem 2.4

The fan graph F_n admitsstar–in–coloring and its star–in–chromatic number is 4, for all odd $n \geq 9$.

Proof: Consider a fan graph $F_n = P_n + K_1$ with $n + 1$ vertices and $2n - 1$ edges. The vertex of K_1 is denoted by v_0 and the remaining vertices are denoted by $v_1, v_2, v_3, \dots, v_n$.
Define a function $f: V \rightarrow \{1, 2, 3, \dots\}$ as below.

We assign $f(v_0) = 4$ *and*

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

By using the above pattern of coloring the fan graph is star–in–colored.

Hence the star–in–chromatic number of F_n is $\chi_{si}(F_n) = 4$.

Illustration 2.4.1 Consider a fan graph F_9, this graph has 10 vertices and 17 edges.

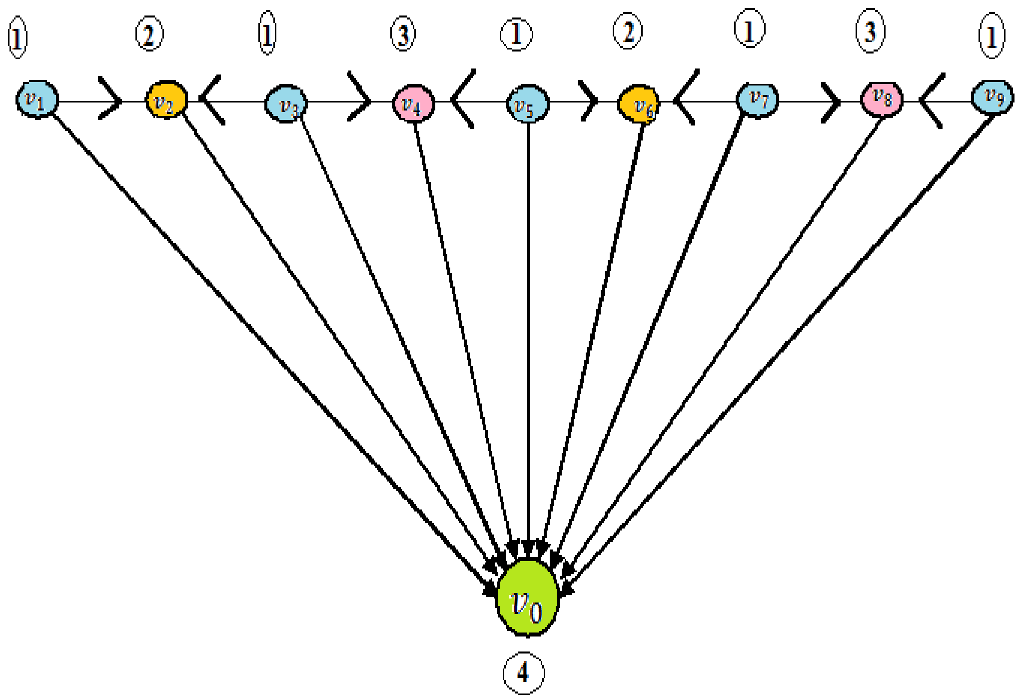


Figure 2.4.1: Fan graph F_9

The vertices v_1, v_3, v_5, v_7 and v_9 are assigned the color 1. The vertices v_2 and v_6 are assigned the color 2. The vertices v_4 and v_8 are assigned the color 3. The vertex v_0 is assigned the color 4.

Hence the star–in–chromatic number of F_9 is $\chi_{si}(F_9) = 4$.

STAR–IN–COLORING OF DOUBLE FAN GRAPH

Theorem 2.5

The double fan graph DF_n admits star-in-coloring and its star–in–chromatic number is 5, for all odd $n \geq 9$.

Proof: Consider a fan graph $DF_n = P_n + \overline{K_2}$ with $n + 2$ vertices and $3n - 1$ edges. The vertices of $\overline{K_2}$ are denoted by u_0, v_0 and the other vertices of DF_n are denoted by $v_1, v_2, v_3, \dots, v_n$.

Define a function $f: V \to \{1, 2, 3, \dots\}$ as follows:

$$f(u_0) = 4\,, f(v_0) = 5 \; and$$

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

By using the above pattern of coloring the double fan graph is star–in–colored and $\chi_{si}(DF_n) = 5$.

Illustration 2.5.1 Consider a double fan graph DF_9. The graph DF_9 consists of 11 vertices and 26 edges.

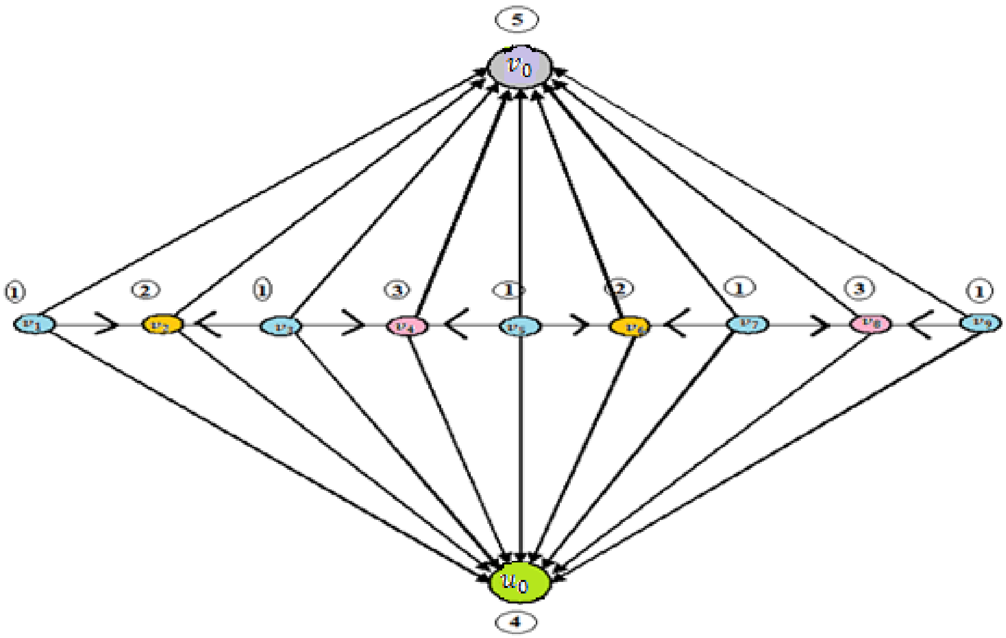


Figure 2.5.1: Double Fan graph DF_9

The vertices v_1, v_3, v_5, v_7 and v_9 are assigned the color 1. The vertices v_2 and v_6 are assigned the color 2. The vertices v_4 and v_8 are assigned the color 3. The vertices u_0 and v_0 are assigned with the colors 4 and 5 respectively.

The star–in–chromatic number of DF_9 is $\chi_{si}(DF_9) = 5$.

STAR–IN–COLORING OF WEB GRAPH

Theorem 2.6

The web graph $W_{n,r}$ admits star–in–coloring and its star–in–chromatic number satisfies the inequality $5 \leq \chi_{si}(W_{n,r}) \leq 7$, for all even n.

Proof: Consider a web graph $W_{n,r}$ this graph consists of nr vertices and $n(2r-1)$ edges. The vertex set V in $W_{n,r}$ are partioned into r vertex sets denoted by $V^1, V^2, V^3, \dots, V^r$ where each vertex set consists of n vertices. The vertex set V^j consists of the vertices $v_1^j, v_2^j, \dots, v_n^j$ for all $1 \leq j \leq r$.

The general pattern of coloring has been grouped into two cases:
One for $n \equiv 0\ (mod\ 4)$ and other for $n \equiv 2\ (mod\ 4)$.

Case 1: $For\ n \equiv 0\ (mod\ 4)$
$f(v_i^j) = 1, if\ i + j$ is even.
For all other values of i and j, we consider four subcases as follows:

Subcase 1.1: $For\ j \equiv 1\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 2, & if\ i \equiv 2 (mod 4) \\ 3, & if\ i \equiv 0 (mod 4) \end{cases}$$

Subcase 1.2:$For\ j \equiv 2\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 4, & if\ i \equiv 1 (mod 4) \\ 5, & if\ i \equiv 3 (mod 4) \end{cases}$$

Subcase 1.3:$For\ j \equiv 3\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 3, & if\ i \equiv 2 (mod 4) \\ 2, & if\ i \equiv 0 (mod 4) \end{cases}$$

Subcase 1.4:$For\ j \equiv 0\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 5, & if\ i \equiv 1 (mod 4) \\ 4, & if\ i \equiv 3 (mod 4) \end{cases}$$

By using the above pattern of coloring the web graph is star–in–colored. According to Case 1 the star–in–chromatic number of $W_{n,r}$ is $\chi_{si}(W_{n,r}) = 5$.

Case 2: $For\ n \equiv 2\ (mod\ 4)$
$f(v_i^j) = 1, if\ i + j$ is even.
For all other values of i and j, we consider four subcases as follows:

Subcase 2.1:$For\ j \equiv 1\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 2, & if\ i \equiv 2 (mod 4)\ and\ i < n \\ 3, & if\ i \equiv 0 (mod 4) \\ 4, & if\ i = n. \end{cases}$$

Subcase 2.2:$For\ j \equiv 2\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 5, & if\ i \equiv 1 (mod 4)\ and\ i < n - 1 \\ 6, & if\ i \equiv 3 (mod 4) \\ 7, & if\ i = n - 1. \end{cases}$$

Subcase 2.3:$For\ j \equiv 3\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 4, & if\ i \equiv 2 (mod 4)\ and\ i < n \\ 2, & if\ i \equiv 0 (mod 4) \\ 3, & if\ i = n. \end{cases}$$

Subcase 2.4: $For\ j \equiv 0\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 7, & if\ i \equiv 1(mod4)\ and\ i < n-1 \\ 5, & if\ i \equiv 3(mod4) \\ 6, & if\ i = n-1. \end{cases}$$

By using the above pattern of coloring the web graph is star–in–colored. According to Case 2 the star–in–chromatic number of $W_{n,r}$ is $\chi_{si}(W_{n,r}) = 7$. From Case(1) and Case(2) the star–in–chromatic number of $W_{n,r}$ satisfies $5 \leq \chi_{si}(W_{n,r}) \leq 7$.

Illustration 2.6.1 Consider a web graph $W_{4,4}$. The graph $W_{4,4}$ consists of 16 vertices and 28 edges.

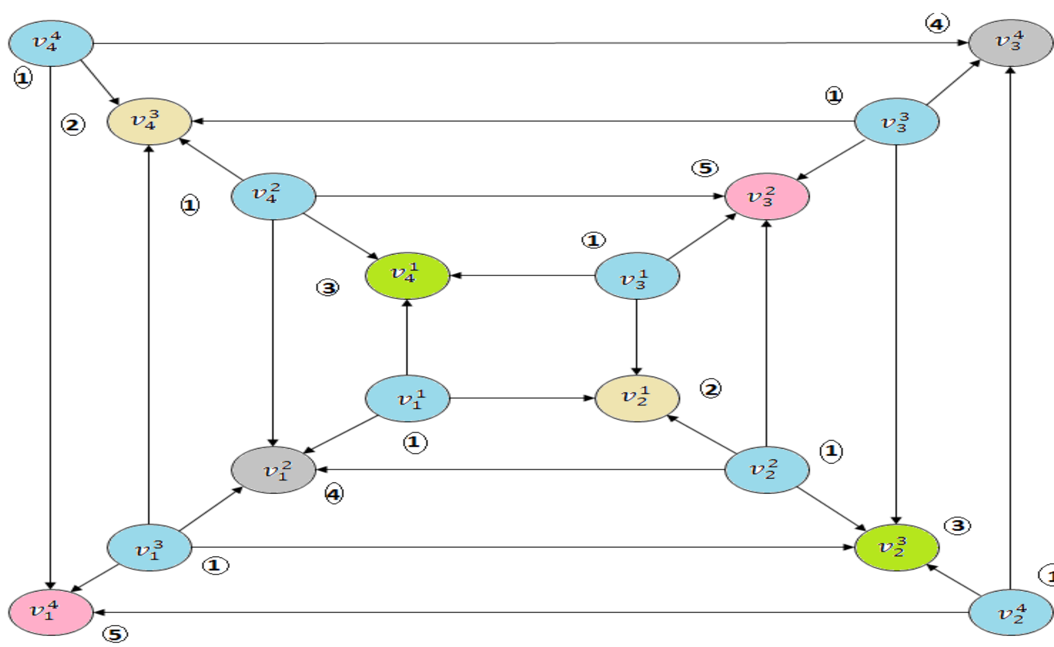


Figure 2.6.1: Web graph $W_{4,4}$.

From Case(1) of Theorem 2.6, the vertices of $W_{4,4}$ are assigned with colors 1, 2, 3, 4 and 5 as shown in Figure 2.6.1 which satisfy the conditions of star–in–coloring.

The star–in–chromatic number of $W_{4,4}$ is $\chi_{si}(W_{4,4}) = 5$.

Illustration 2.6.2 Consider a web graph $W_{6,4}$. This graph consists of 24 vertices and 42 edges.

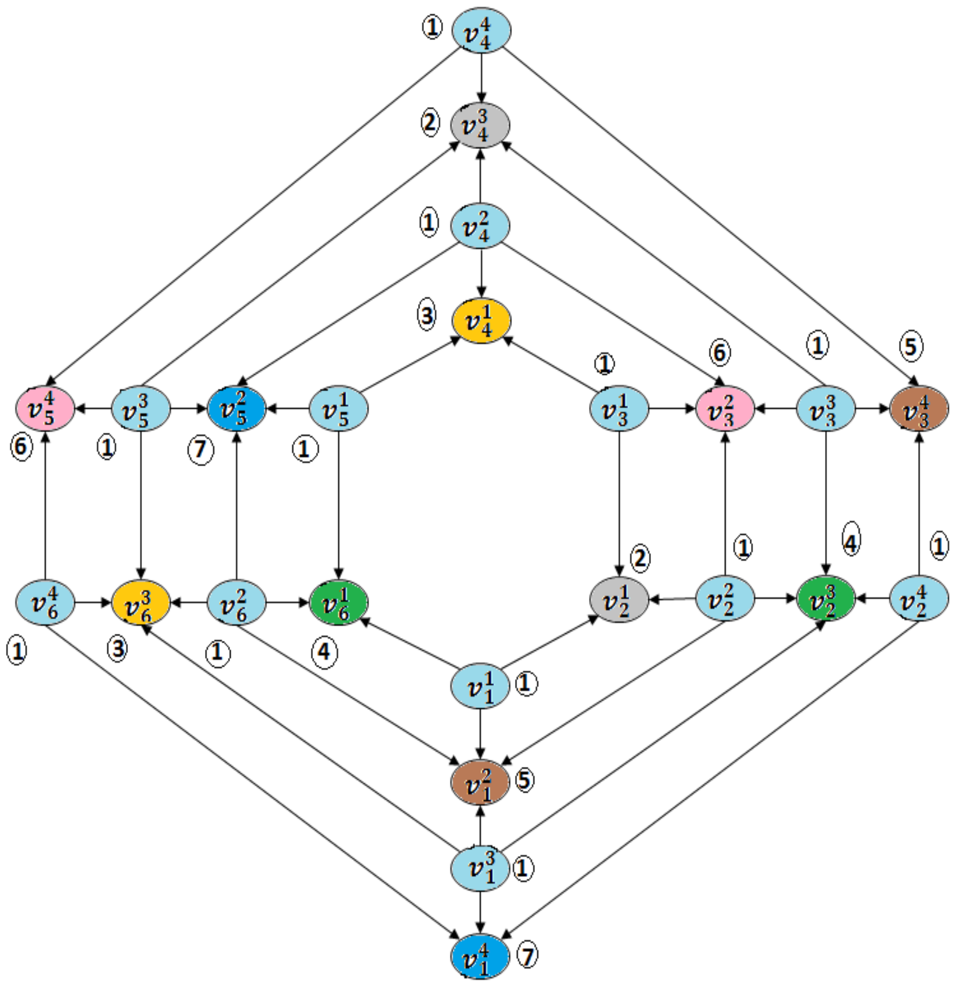


Figure 2.6.2: Web graph $W_{6,4}$.

From Case(2) of Theorem 2.6, the vertices of $W_{6,4}$ are assigned with colors 1, 2, 3, 4, 5, 6 and 7 as shown in Figure 2.6.2 which satisfy the conditions of star–in–coloring.

The star–in–chromatic number of $W_{6,4}$ is $\chi_{si}(W_{6,4}) = 7$.

STAR–IN–COLORING OF COMPLETE BINARY TREE

Theorem 2.7

The complete binary tree BT_n admitsstar–in–coloring and its star–in–chromatic number is 3, for all $n \geq 2$.

Proof: Consider a complete binary tree BT_n with $|V| = 1 + 2 + 2^2 + \cdots + 2^n$ vertices and $|E| = |V| - 1$ edges. The root vertex (degree 2) is denoted by v_0 and the other vertices are denoted by $v_1^1, v_1^2, v_2^1, v_2^2, \ldots$

We define $f : V \to \{1, 2, 3, \ldots\}$ as follows.

$$\text{We assign } f(v_0) = 1 \text{ and}$$

$$f\left(v_i^j\right) = \begin{cases} 1, & if\ i \equiv 0(mod3) \\ 2, & if\ i \equiv 1(mod3) \\ 3, & if\ i \equiv 2(mod3) \end{cases}$$

With this pattern of coloring, the complete binary tree BT_n can be star–in–colored and $\chi_{si}(BT_n) = 3$ for all n ≥ 2.

Illustration 2.7.1 Consider the complete binary tree BT_2, which consists of 7 vertices and 6 edges.

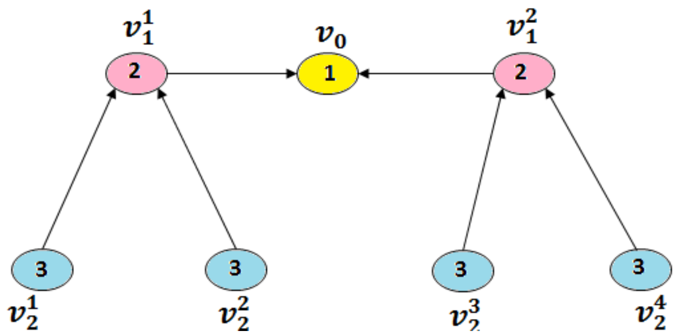


Figure 2.7.1: Complete Binary Tree BT_2.

By Theorem 2.7, the vertices v_1^1 *and* v_1^2 are assigned the color 2. The vertices v_2^1, v_2^2, v_2^3 *and* v_2^4 are assigned the color 3. The root vertex v_0 is assigned the color 1.

The star–in–chromatic number of BT_2 is $\chi_{si}(BT_2) = 3$.

3. CONCLUSION

In this paper, we have obtained the lower and upper bounds for star–in–chromatic number of some of the standard graphs as below.

1. $3 \leq \chi_{si}(C_n) \leq 4, n$ is even.
2. $\chi_{si}\big(RC(p,n)\big) = n + 1, p > 3$.
3. $4 \leq \chi_{si}(G_n) \leq 5, n \geq 4$.
4. $\chi_{si}(F_n) = 4$, odd $n \geq 9$.
5. $\chi_{si}(DF_n) = 5$, odd $n \geq 9$.
6. $5 \leq \chi_{si}\big(W_{n,r}\big) \leq 7, n$ is even.
7. $\chi_{si}(BT_n) = 3, n \geq 2$.

REFERENCES

1. Alon.N, McDiarmid.C, and Reed.B. Acyclic colourings of graphs. *Random Structures and Algorithms*, 2:277–288, (1990).
2. Borodin.O.V, Kostochka.A.V, and Woodall.D.R. Acyclic colourings of planar graphswith large girth. *J. London Math. Soc.*, 60 (2):344–352, (1999).
3. Borodin.O.V, Kostochka.A.V, Raspaud.A, and Sopena.E. Acyclic k-strong coloring of maps on surfaces. *Mathematical Notes*, 67 (1):29–35, (2000).
4. Coxter. H. S. M, (1950) Self-dual configurations and regular graphs, *Bulletin of the American Mathematical Society,* 56, 413-455.
5. Fertin. G, Raspaud. A, Reed. B, (2001), On star coloring of graphs, *Graph Theoretic Concepts in Computer Science,* 27th International Workshop, WG 2001, Springer Lecture Notes in Computer Science, 140-153 (2204).
6. Frank Harary, *Graph Theory* (Narosa Publishing House, 2001).
7. Grünbaum. B, Acyclic colorings of planar graphs, *Israel J. Math.*14, 390-408 (1973).
8. Kim.D.S, Du.D.Z, and Pardalos.P.M. A coloring problem on the n-cube. *Discrete Applied Mathematics,* 103:307–311, (2000).
9. Nesetril. J and Ossona de Mendez. P, Colorings and homomorphisms of minor closed classes, *Discrete and Computational Geometry:* The Goodman Pollack Festschrift (ed. B. Aronov, S. Basu, J. Pach, M. Sharir), Springer Verlag, 651-664 (2003).
10. Sudha. S and Kanniga. V, Star-in-coloring of Complete bi-partite graphs, Wheel graphs and Prism graphs, *International Journal of Engineering and Technology,* Vol 2, Issue 2, 97-104 (2014).