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Domination, Equitable and End Equitable Domination Numbers of Some Graphs

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ABSTRACT

A subset D of $V(G)$ is called an *equitable dominating set* of a graph G if for every $u \in (V-D)$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|\deg(u)-\deg(v)| \leq 1$. An equitable dominating set D is said to be an *end equitable dominating set* of G if D contains all the end vertices of graph G. In this paper, we discuss the domination, equitable domination and end equitable domination numbers of the graphs such as lollipop $L_{m,n}$, butterfly $BF(m,n)$, jellyfish $J(m,n)$ and subdivision of jellyfish $S(J(m,n))$.

AMS Mathematics Subject Classification (2000): 05C50.

Keywords: Domination number; equitable domination number; end equitable domination number.

1. INTRODUCTION

All the graphs considered here are finite, undirected with no loops and multiple edges. A *graph* G consists of a pair (V,E) where V is a non-empty finite set whose elements are called *vertices or nodes* and E is a set of unordered pairs of distinct elements of V. The elements of E are called *edges* of the graph G. The *degree* of a vertex of a graph is the number of edges incident to the vertex, with loops counted twice. The degree of a vertex v is denoted by $\deg(v)$. The maximum and minimum degree of a graph is denoted by $\Delta(G)$ and

¢(G) respectively. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v respectively. A vertex $v \in G$ is called *pendant vertex* or end vertex of G if $\deg(v) = 1$. An edge of a graph is said to be *pendant* if one of its vertices is a pendant vertex. For graph theoretic terminology we refer to Chartrand and Lesniak[4].

The rigorous study of dominating sets in Graph theory began around 1960. According to Hedetneimi and Laskar (1990)[8], the domination problems were studied from the 1950's onwards, but the rate of research on domination significantly increased in the mid 1970's. In 1958, Berge[1] defined the concept of the domination number of a graph called as "coefficient of external stability". In 1962, Ore[15] coined the name "dominating set" and "domination number" for the same concept. In 1977 Cockayane and Hedetneimi[5] made an interesting and extensive survey of the results known at the time about dominating sets in graphs[7,14]. They have used the notation ¥(G) for the domination number of a graph, which has become very popular since then[3]. The survey paper of Cockayane and Hedetniemi[5] has generated lot of interest in the study of dominating sets in graphs. Recent books on domination, have stimulated a sufficient inspiration leading to the expansive growth of this field to study. The domination number for the helm graph H_n and web graph W_n were found by Ayhan A. Khalil[10]. The domination and equitable domination number for the friendship graph F_n and windmill graph $Wd(m,n)$ were proved by Dr. C S Nagabhushana *et al*.[13]. The domination and equitable domination number for the book graph B_n and stacked book graph $B_{3,n}$ were proved by Kavitha B N and Indrani Kelkar[9].

The concept of equitable domination number in graphs was introduced by Swaminathan *et al*.[16] by considering the following real world problems such as network nodes with nearly equal capacity may interact with each other in a better way, in our society persons with nearly equal status, tend to be friendly, in an industry, employees with nearly equal powers form association and move closely, equitability among citizens in terms of wealth, health, status etc is the goal of a democratic nation. The end equitable domination number in graph has introduced by J.H.Hattingh and M.H.Henning[6]. The end equitable domination results are proved by K. B. Murthy and Puttaswamy[11,12].

2. DEFINITIONS AND NOTATIONS

In this section we recall some of basic definitions in literature which will be useful for our present work.

Definition 2.1:[2,17] A set D of vertices in a graph $G = (V, E)$ is called a *dominating set* of G, if every vertex in $V - D$ is adjacent to some vertex in D. The *domination number* ¥(G) of a graph G is the minimum cardinality of the dominating set in G.

Definition 2.2: A subset D of $V(G)$ is called an *equitable dominating set* of a graph G if for every $u \in (V - D)$; there exists a vertex $v \in D$ such that $uv \in E(G)$ and

$|\deg(u)-\deg(v)|\leq 1$. The minimum cardinality of such a *dominating set* is denoted by ¥$_e(G)$ and is called *equitable domination number* of G.

Definition 2.3: If a vertex $u \in V$ be such that $|\deg(u)-\deg(v)| \geq 2$ for all $v \in N(u)$, then u is called an *equitable isolate vertex*.

Definition 2.4: An equitable dominating set D is said to be an *end equitable dominating set* of G if D contains all the end vertices of graph G. The minimum cardinality of an end equitable dominating set is called the *end equitable domination number* of G and is denoted by ¥$_{ee}(G)$.

Definition 2.5: A *lollipop* graph is the graph obtained from a complete graph K_m with one vertex is attached by a path of length n and it is denoted by $L_{m,n}$.

Definition 2.6:[23] A *shell* S_n is the graph obtained by taking $(n-3)$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex.

Definition 2.7:[23] A *bow* graph is a double shell (consists of two edge disjoint shells with a common apex) in which each shell has any order.

Definition 2.8:[23] A *butterfly* graph is a bow graph with exactly two pendant edges at the apex vertex.

Definition 2.9: A G is a graph that can be obtained from G by subdivision of each edge of G is called the *subdivision* of a graph G and it is denoted by $S(G)$.

Definition 2.10: [22] The *jellyfish* graph $J(m,n)$ is obtained from a 4-cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendant edges to v_2 and n pendant edges to v_4.

Definition 2.11: The *floor function* of a real number x is the greatest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$. Suppose that $n \leq x < n+1$, where n is an integer, then $\lfloor x \rfloor = n$.

Definition 2.12: The *ceiling function* of a real number x is the lowest integer greater than or equal to x and it is denoted by $\lceil x \rceil$. Suppose that $n-1 < x \leq n$, where n is an integer, then $\lceil x \rceil = n$.

3. DOMINATION, EQUITABLE DOMINATION AND END EQUITABLE DOMINATION NUMBERS

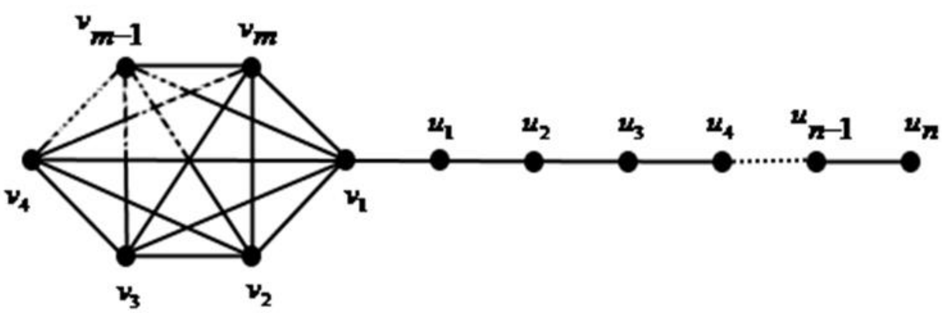
Theorem 3.1. For any lollipop graph $L_{m,n}$, the domination number is $1+\left\lceil \frac{n-1}{3} \right\rceil$, where $m \geq 2, n \geq 1$.

Proof. Let $G \cong L_{m,n}$ be a lollipop graph on $m+n$ vertices and $\left[\frac{m(m-1)}{2} \right] + n$ edges where $m \geq 2, n \geq 1$. Note that G has no equitable isolated vertex. Any minimum dominating set of G must contain the vertex v_1, since $\deg(v_1) = \Delta(G)$.

To minimize the size of the dominating set we will select v_1 as one element.

The vertex v_1 will dominate the vertices u_1 and $\{v_2, v_3, ..., v_m\}$. We know that the domination number of a complete graph is one.

Fig. 3.1.



Lollipop graph $L_{m,n}$

That is ¥$(K_m) = 1$.

Hence ¥$(G) =$ ¥$(K_m) +$ ¥(P_{n-1})

¥$(G) = 1 + \left\lceil \frac{n-1}{3} \right\rceil$.

Theorem 3.2. For any lollipop graph $L_{m,n}$, the equitable domination number is $1+\left\lceil \frac{n}{3} \right\rceil$, where $m \geq 4, n \geq 1$.

Proof. Let $G \cong L_{m,n}$ be a lollipop graph.

Any two vertices in K_m, where $m \geq 3$ are equitable dominating vertices and any two adjacent vertices in P_n, where $n \geq 1$ are also equitable dominating vertices.

In G, there is no equitable isolated vertex and $\deg(v_1) = \Delta(G)$. Any minimum dominating set of G must contain the vertex v_1. However v_1 is not equitable dominating with u_1.

From Fig. 3.1, the vertex v_1 will dominate the vertices in the set $\{v_2, v_3, ..., v_m\}$.

Hence $¥_e(G) = ¥(K_m) + ¥(P_n)$

$$¥_e(G) = 1 + \left\lceil \frac{n}{3} \right\rceil.$$

Theorem 3.3. For any lollipop graph $L_{m,n}$, the end equitable domination number is $2 + \left\lceil \frac{n-2}{3} \right\rceil$, where $m \geq 4, n \geq 1$.

Proof. Let $G \cong L_{m,n}$ be a lollipop graph.

Any two vertices in K_m, where $m \geq 3$ are equitable dominating vertices and any two adjacent vertices in P_n, where $n \geq 1$ are also equitable dominating vertices.

By definition of end equitable dominating set D, the set D contains the end vertex u_n. Further $\deg(v_1) = \Delta(G)$, then any minimum end equitable dominating set must contain the vertices v_1 and u_n. v_1 will dominate the vertices in the set $\{v_2, v_3, ..., v_m\}$ and u_n will dominate the vertex u_{n-1}.

$$¥_{ee}(G) = ¥(K_m) + ¥(P_{n-2}) + |\{u_n\}| = 1 + \left\lceil \frac{n-2}{3} \right\rceil + 1$$

Hence $¥_{ee}(G) = 2 + \left\lceil \frac{n-2}{3} \right\rceil$.

Example 3.4. The lollipop graph $L_{4,5}$ is shown in Fig 3.2.

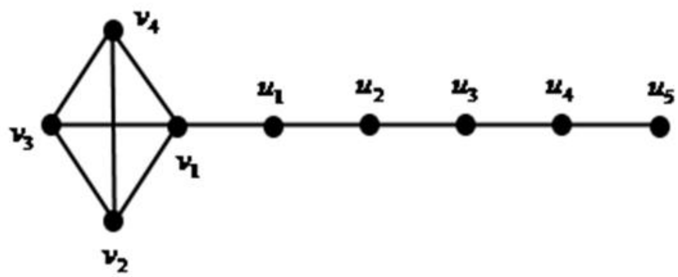


Fig. 3.2. Lollipop graph $L_{4,5}$

In this graph, the minimum dominating set is $D=\{v_1,u_3,u_5\}=3$, the minimum equitable dominating set is $D=\{v_1,u_2,u_4\}=3$ and minimum end equitable dominating set is $D=\{v_1,u_2,u_5\}=3$ and so $¥(G)=¥_e(G)=¥_{ee}(G)=3$.

Remarks 3.5. In the case of lollipop graph $L_{m,n}$, when $m=2, n\geq 1$, the minimum dominating and minimum equitable dominating set are the same. i.e., $¥(G)=¥_e(G)=\left\lceil \frac{n+2}{3} \right\rceil$ and the minimum end equitable dominating set is $¥_{ee}(G)=2+\left\lceil \frac{n-2}{3} \right\rceil$. If $m=3, n\geq 1$, the minimum dominating, minimum equitable dominating and minimum end equitable dominating sets of lollipop graph $L_{3,n}$ and tadpole graph $T_{3,n}$ are the same.

Theorem 3.6. For any butterfly graph $BF(m,n)$, the domination number is 1, the equitable and end equitable domination number is $\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + 3$, where $m,n\geq 1$.

Proof. Let G be a butterfly graph with shells of orders m and n excluding the apex and it has $m+n+3$ vertices with $2(m+n)$ edges. The apex of the butterfly graph is denoted as $'u'$, we denote the vertices in the left wing and right wing of butterfly graph from top to bottom by $v_1,v_2,...,v_m$ and $u_1,u_2,...,u_n$ and the pendant vertices in the pendant edges are denoted by v and w. Clearly G has two end vertices v and w.

In any minimum dominating set $'u'$ must be included, since $\deg(u)=\Delta(G)$. Also, $N(u)=\{u_1,u_2,...,u_n,v_1,v_2,...,v_m,v,w\}$.

Hence $D=\{u\}$ is the minimum dominating set of G.

$\therefore ¥(G)=1$.

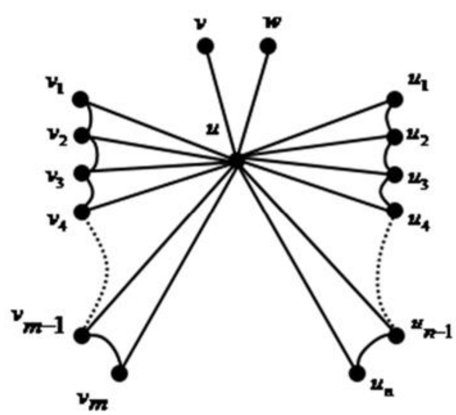


Fig. 3.3. Butterfly graph $BF(m,n)$

Since any two vertices in the set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_m\}$ are also equitable dominating vertices in both right and left wings. Clearly, the vertices u, v and w are equitable isolated vertices. So these vertices must be included in any minimum equitable dominating set. For any end equitable dominating set D, the end vertices v and w must be included in D.

Therefore $¥_e(G) = ¥_{ee}(G) = [¥_e(P_m) + ¥_e(P_n)] + |\{u, v, w\}|$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + 3$$

Hence $¥_e(G) = ¥_{ee}(G) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + 3$.

Example 3.7. The butterfly graph $B(5,4)$ is shown in Fig. 3.4.

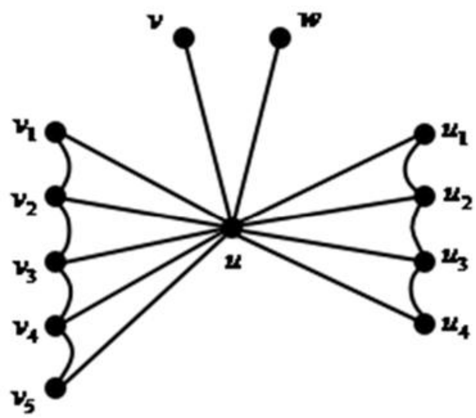


Fig. 3.4. Butterfly graph $BF(5,4)$

In this graph, the minimum dominating set is $D = \{u\} = 1$ and so $¥(G) = 1$. The minimum equitable dominating and minimum end equitable dominating set is $D = \{u, v, w, v_2, v_5, u_2, u_4\} = 7$. Note that $¥_e(G) = ¥_{ee}(G) = \left\lceil \frac{5}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil + 3 = 7$.

Theorem 3.8. For any jellyfish graph $J(m,n)$, the domination number is 2, the equitable and end equitable domination number is $m+n+3$, where $m, n \geq 3$.

Proof. Let $G \cong J(m,n)$ be a jellyfish graph on $m+n+4$ vertices and $m+n+5$ edges, where $m, n \geq 3$.

The set $D = \{u, v\}$ is a minimum dominating set for the graph G, that means the set D will dominate all the other vertices in G.

$\therefore ¥(G) = 2$.

Let $E=\{u_1,u_2,...,u_m,v_1,v_2,...,v_n\}$. The set E consists of all end vertices of G. Hence any minimum end equitable dominating set must contains the set E. Further, each vertex in E is an equitable isolated vertex, so any minimum dominating set also contains the set E.

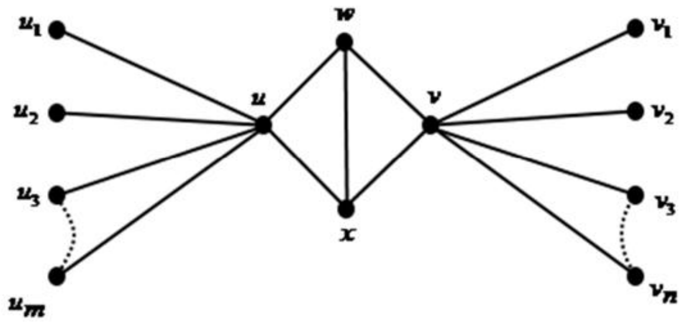


Fig. 3.5. Jellyfish graph $J(m,n)$

Further $\deg(u)=\deg(v)=\Delta(G)$, so the vertices u and v must be included in any minimum equitable and minimum end equitable dominating sets. Moreover, the vertices x and w are equitable dominating vertices of each other. Hence anyone of the vertices x or w must be included in any minimum equitable as well as end equitable dominating sets.

$\therefore ¥_e(G)=¥_{ee}(G)=\left|E\cup\{u,v,w\}\right|=m+n+3$.

Hence $¥_e(G)=¥_{ee}(G)=m+n+3$.

Example 3.9. The jellyfish graph $J(5,4)$ is shown in Fig. 3.6.

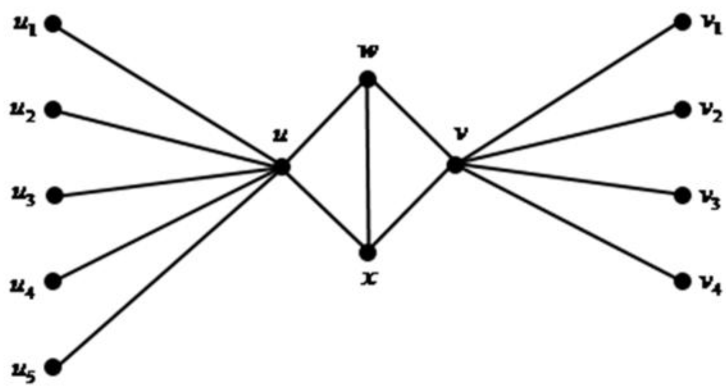


Fig. 3.6. Jellyfish graph $J(5,4)$

In this graph, the minimum dominating set $D=\{u,v\}=2$ and so $¥(G)=2$, the minimum equitable dominating and minimum end equitable dominating set is $D=\{u,v,w,u_1,u_2,u_3,u_4,u_5,v_1,v_2,v_3,v_4\}=12$ and so $¥_e(G)=¥_{ee}(G)=12$.

Remarks 3.10. In Jellyfish graph $J(m,n)$, if $1 \le m,n \le 2$, then the vertices x or w equitable dominating to all its adjacent vertices. Hence
$¥_e(G) = ¥_{ee}(G) = m + n + \left|\{w\}\right| = m + n + 1$. If $m \ge 3$, $1 \le n \le 2$, then the vertex u must be included in any minimum equitable as well as end equitable dominating set. Hence $¥_e(G) = ¥_{ee}(G) = m + n + \left|\{u, w\}\right| = m + n + 2$. This is hold for the case $n \ge 3$, $1 \le m \le 2$ also.

Theorem 3.11. For a subdivision of jellyfish graph $S(J(m,n))$, the domination number is $m + n + 3$, the equitable and end equitable domination number of $S(J(m,n))$ is $m + n + 4$, where $m,n \ge 2$.

Proof. Let G be the subdivision of jellyfish graph $J(m,n)$ and it has $2m + 2n + 9$ vertices and $2m + 2n + 10$ edges.

In any dominating, equitable dominating and end equitable dominating sets, the vertices u, v must be included, since $\deg(u) = \deg(v) = \Delta(G)$. Out of the vertices w, e and x, the vertex e dominates both w and x.

Hence the set $D = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, u, v, e\}$ is a minimum dominating set for the graph G, that means the set D will dominate all the other vertices in G.

$\therefore ¥(G) = m + n + 3$

Now the set of equitable isolated vertices $E = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ must be included in any minimum equitable and minimum end equitable dominating set and these vertices will equitable dominate the set of vertices as $\{u_1', u_2', ..., u_m', v_1', v_2', ..., v_n'\}$. If we choose the vertices w and x, it will dominate the vertices a, b, c, d and e.

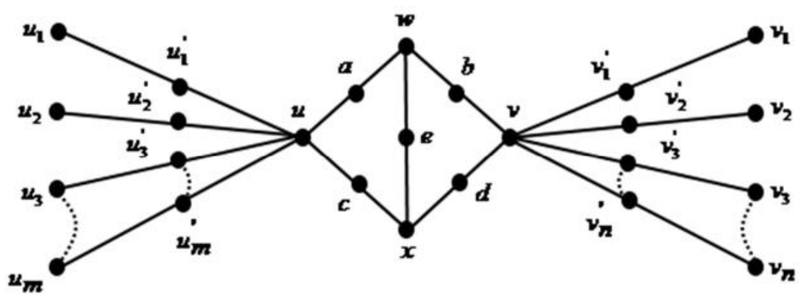


Fig. 3.7. Subdivision of jellyfish graph $S(J(m,n))$

Hence the set $E \cup \{u, v, w, x\}$ is a minimum equitable and minimum end equitable dominating set, since u and v are equitable isolated vertices. So $\deg(u) = \deg(v) = \Delta(G)$.

$\therefore ¥_e(G) = ¥_{ee}(G) = \left|E \cup \{u, v, w, x\}\right| = (m + n) + 4 = m + n + 4$.

Hence $¥_e(G) = ¥_{ee}(G) = m + n + 4$.

Example 3.12. The subdivision of jellyfish graph $S(J(4,3))$ is shown in Fig. 3.8.

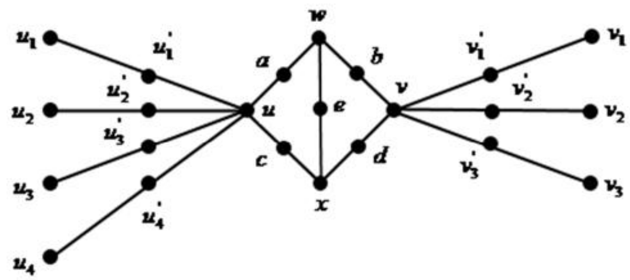


Fig. 3.8. Subdivision of jellyfish graph $S(J(4,3))$

In this graph, the minimum dominating set is $D = \{u, v, e, v_1, v_2, v_3, u_1, u_2, u_3, u_4\} = 10$ and so $¥(G) = 10$, the minimum equitable dominating and end equitable dominating set is $D = \{u, v, w, x, v_1, v_2, v_3, u_1, u_2, u_3, u_4\} = 11$ and so $¥_e(G) = ¥_{ee}(G) = 11$.

Remarks 3.13. In the case of subdivision of jellyfish graph $J(m, n)$, when $m, n = 1$, the minimum dominating, minimum equitable dominating and minimum end equitable dominating sets are the same. i.e., $D = \{u, v, e, u_1, v_1\}$ is the domination, equitable domination and end equitable domination number is $¥(G) = ¥_e(G) = ¥_{ee}(G) = 5$.

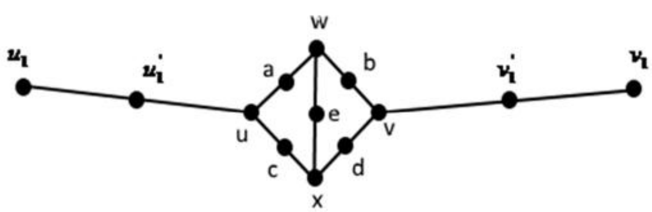


Fig. 3.9. Subdivision of jellyfish graph $S(J(1,1))$

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