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One Dimensional Topological Quantum Field Theory

A. Ramesh Kumar[1] and R. Prabhu[2]

[1]Assistant Professor, Department of Mathematics,
Srimed Andavan Arts & Science College, Trichy-05, INDIA.
2Assistant Professor, Department of Mathematics,
M.I.E.T Engineering College, Trichy-07, INDIA.
email:andavanmathsramesh@gmail.com., prabhu.rajamanikam@gmail.com.

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ABSTRACT

Topological quantum field theory by motivating its axioms through path integral considerations. The resulting description in terms of symmetric monodial functors from bordisms to vector spaces is introduced and some of its immediate consequences are expounded on, still in general dimension n. To generators-and-relations descriptions of topological quantum field theories, which allow one to cast their study into algebraic language. The resulting algebraic structure is presented in some detail for dimension one.

Keywords: Topological quantum field, symmetric monodial functors, generators-and-relations.

1. INTRODUCTION

Topological quantum field theories are a rewarding area of study in mathematical physics and pure mathematics. They appear in the description of physical systems such as the fractional quantum Hall effect; they are used in topological quantum computing; they are important renormalization group flow invariants of supersymmetric field theories obtained via“twisting”; they have a clean mathematical axiomatisation; they give invariants of knots and of manifolds; they play vital roles in mirror symmetry and the geometric Langlands Programme.

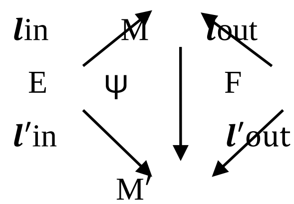
The quantum field theory from an angle where it appears as a way of transporting the geometric and dynamical structure of spacetime into the algebraic description of physical states and observables. From the functional perspective a quantum field theory is a map

$$\text{‘geometry’} \rightarrow \text{’algebra’} \qquad (1)$$

That preserves certain structure. It is a difficult and important problem to make this precise for general quantum field theories. For the special case when"geometry" basically only means "topology"**.** The other symmetric monoidal category we need is $Bord_n$, which is named after its morphisms. Objects in $Bord_n$ are oriented closed (n-1)-dimensional real manifolds E for some fixed$n \in \mathbb{Z}_{\geq 1}$.we may think of E as a toy model of a special slice of n-dimensional spacetime. A morphisms E→F in $Bord_n$ is an equivalence class of a bordism from E to F.A bordism E→F is an oriented compact n-dimensional manifold M with boundary, together with smooth mapsι_{in}:E→M←F: ι_{out} with image in ∂M such that

$$\iota_{in} \sqcup \iota_{out}: \bar{E} \sqcup F \to \partial M \tag{2}$$

is an orientation – preserving diffeomorphism, where $\bar{E}$ denotes F with the opposite orientation. Two bordisms (M, ι_{in}, ι_{out}),$(M', \iota'_{in}, \iota'_{out})$: E→F are equivalent if there exists an orientation-preserving diffeomorphism $\psi: M \to M'$ such that commutes.



This is how the smooth geometric structure is discarded in $Bord_n$.Comosition of morphisms M_1:E→F and M_2: F→G in $Bord_n$ is given by "gluing M_1 and M_2 along F".

Apart from studying concrete example of topological quantum field theories, it is an important question to what extent one can control all topological quantum field theories of a given dimension. One might formulate this goal as "classification of topological quantum field theories", but in a sense this is a bad term to use. For, if some hands you a paper saying it contains the classification of topological quantum field theories in some dimension n, you might hope for some sort of list,e.g.one that says that there are such infinite families parametrised by this and thatset, plus a bunch of exceptional. However, for topological quantum field theories bounded dimension n=1, this is impossible, much in the same way that there cannot be a list of "all finite groups".

The roadmap for this section we discuss at length how one dimensional topological quantum field theories are basically finite-dimensional vector spaces-only to rephrase and refine this discussion where we introduce the notion of freely generated symmetric monoidal categories. Then examines two dimensional topological quantum field theories through this lens, finding how such topological quantum field theories are in one-to-one correspondence to commutative Frobenius algebras.

Definition 1.1

An n-dimensional oriented closed topological quantum field theory is a symmetric monoidal functor

$$\mathbb{Z}: Bord_n \to Vect_k, \tag{4}$$

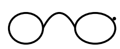
We will unravel this statement by discussing the two categories $\mathrm{Bord_n}$ and $\mathrm{Vect_k}$. Highlighting the key properties of the functor Z, and indicating how it encodes the structure motivated for a neat technical review of symmetric monoidal categories, functors and their natural transformations.

Proposition 1.2

Let$\mathbb{Z}: Bord_n \rightarrow\ Vect_k$, be a topological quantum field theory. Then Z(E) is finite dimensional for every E$\in Bord_n$, and $\overline{Z(E)} \cong Z(E)*$.

Proof: The origin of this finiteness property is duality. To see this, let us set $U: Z(E)$ and V: $Z\overline{(E)}$

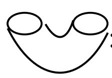
For the vector space associated to the manifold with opposite orientation. Next we consider the cylinder E× [0, 1], but viewed as a morphism

 $\bar{E}\sqcup E \rightarrow \theta.$ (5)

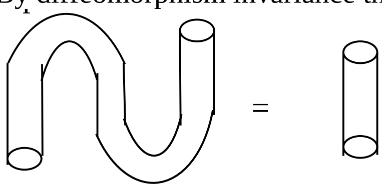
$\bar{E}$ E

Similarly, we can view the cylinder as a map

$\bar{E}$ E

: $\theta \rightarrow \bar{E}\sqcup E.$ (6)

By diffeomorphism invariance these two maps are related to the identity 1_E as follows:



(7)

Applying Z, we obtain a pairing $< -,->: Z($  $): V\otimes_k U \rightarrow k$, a copairing γ: Z () : k → $U\otimes_k V$,and translates into the identity

$(< -,-> \otimes idv) o (idv \otimes \gamma) = idv.$ (8)

We may choose finitely many $u_i\ \in U\ and\ v_i\ \in V$ such that$\gamma(1) = \sum_i u_i \otimes v_i$. Using this in (8) we find that for every $v\ \in V$

$$v = \sum_i < v, u_i >. v_i \quad (9)$$

Which proves that the finite set $\{v_i\}$ spans V, so Z (E) is indeed finite-dimensional. Furthermore, one checks that V→U*, v→$< v, ->$ is an isomorphism.

2. ONE-DIMENSIONAL TOPOLOGICAL QUANTUM THEORIES

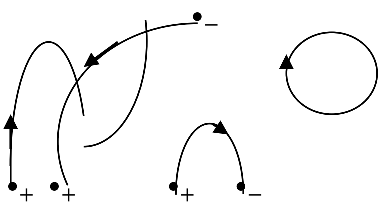
We start with the case of a one-dimensional topological quantum theory

$$\mathbb{Z}: Bord_1 \rightarrow\ Vect_k, \quad (10)$$

We first note that $Bord_1$ is "tensor-generated" by just two objects: the positively and negatively oriented points $\bullet_+$ and$\bullet_-$, respectively, this is simply another way of saying that every 0-dimensional compact oriented closed manifold is a disjoint union of finitely many copies of $\bullet_+$ and $\bullet_-$.Hence the objects of $Bord_1$ look like

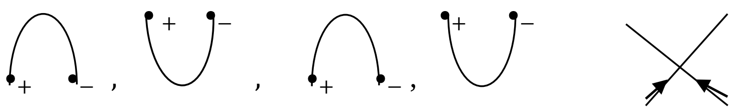
$$\emptyset,\ \bullet_+,\ \bullet_-\sqcup\bullet_-,\ \bullet_+\sqcup\bullet_+\sqcup\bullet_-\sqcup\ldots\ldots\ldots\ldots\sqcup\bullet_+. \qquad (11)$$

Morphisms in $Bord_1$ are diffeomorphism classes of oriented lines connecting such points. For example, one particular morphism from $\emptyset$, $\bullet_+\sqcup\ \bullet_-\ \sqcup\bullet_+\sqcup\bullet_-\sqcup$ to $\bullet_+\sqcup\bullet_-$ is represented by $\bullet_+$



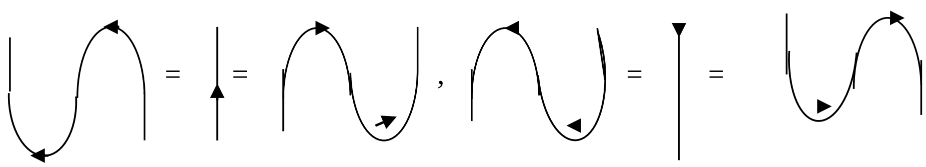
(12)

It is an unsurprising fact that every morphism in $Bord_1$ can be built by composing the identity $1_{\bullet_+} = \uparrow$ and the generators



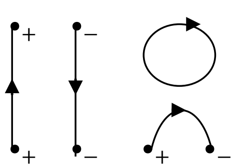
(13)

Where the last diagram represents the braiding bordism $\beta_{\bullet_+,\bullet_+}$. Note that typically we do not show oriented endpoints in such diagrams. The generators (13) are subject to the relations



(14)

As well as further relations involving the braiding, namely the hexagon equations and a relation that expresses the naturality of $\beta_{\bullet_+,\bullet_+}$.for example, these relations tell us that morphisms (12) is equivalently represented by the bordism



(15)

Since with the above generators and relations $Bord_1$ is completely under control, one can give preliminary "classification of" one-dimensional topological quantum field theories.

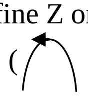
Theorem 2.1

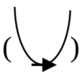
There is a 1-to-1 correspondence between one-dimensional topological quantum field theories. $\mathbb{Z}: Bord_n \rightarrow\ Vect_k$ and finite-dimensional vector spaces, given by Z→Z ($\bullet_+$).

It follows from Proposition (1.1) that Z ($\bullet_+$) is indeed finite-dimensional. Conversely, for every finite-dimensional vector space V we construct a symmetric monoidal functor Z: $\mathbb{Z}: Bord_1 \rightarrow\ Vect_k$ as follows: set Z ($\bullet_+$)=V and Z ($\bullet_-$)=V*, and more generally

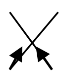
$$Z\ (\bullet_+^{\sqcup m} \sqcup \bullet_-^{\sqcup n}) = V^{\otimes m} \otimes_k (V\ *)^{\otimes n} \tag{16}$$

To define Z on generators we pick a basis $\{e_i\}$ of V set

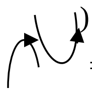
Z () : $V^{*} \otimes_k V \rightarrow K, \varphi \otimes v \rightarrow \varphi(v),$

Z () : $k \rightarrow V \otimes\ _k V(v\ *, \lambda \rightarrow \sum_i \lambda.\, e_i \otimes e *_i,$

Z () : $V^{*} \otimes_k V \rightarrow K,\ \lambda \rightarrow \sum_i \lambda.\, e *_i \otimes e_i,$

Z () : $V \otimes_k V \rightarrow\ V \otimes_k V$, u ⊗ v → v ⊗ u (17)

It is straightforward to verify that Z really is a symmetric monoidal functor, i.e. that it respects the relations (14) for the second identity, say, we compute

Z () = (v→ (Z (   ) ⊗ id) o (id ⊗ Z (   )) (v⊗1)

= (v→ (Z () ⊗ id) o (v$\otimes \sum_i e_i^* \otimes e_i$))

= (v → $\sum_i e_i^*\ (v) \otimes e_i$)

= Z (

And the other relations are checked similarly. This establishes the 1-to-1correspondance of above result.

The upshot so far is that one-dimensional topological quantum field theories are boring: finite-dimensional vector spaces with no further structure. So they are basically natural numbers.

The reason why the above way of stating above theorem is preliminary is that “1-to-1 correspondence” is not really a term one should use when comparing categories. Then topological quantum field theories of a given dimension from a groupoid. So really we would like to have a statement like this:

$$\begin{pmatrix} groupoid\ of \\ n - dimensional\ TQFTs \end{pmatrix} \frac{functorial}{equivalence} \begin{pmatrix} some\ algebraic\ structure \\ which\ also\ forms\ a\ groupoid \end{pmatrix}$$

The algebraic structure suggested by our preliminary theorem is finite-dimensional vector spaces. However, the natural way of comparing vector spaces are linear maps, and these do not form a groupoid. Of course, one can just throw out all non-invertible linear maps, but the

Structural purist will insist that firstly, one would then have failed to identity the correct algebraic structure, and secondly, there is no inherent reason to give preference to $\bullet_+$ over $\bullet_-$.

So, let us instead describe the category DP_k of dual pairs.

3. GENERATORS AND RELATIONS

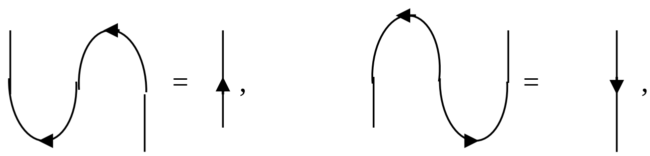
Before we continue to two dimensions, we spend a little effort to make precise the phrase "freely generated as a symmetric monoidal category" As our run-along example we take the one-dimensional topological quantum field theories from above. In particular, we will see that $Bord_1$ is freely generated as a symmetric monoidal category by the objects

$\bullet_+ , \bullet_-$ (18)

And the morphisms

(19)

Subject to the relations



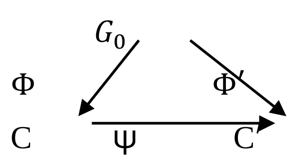
(20)

3.1 Objects only

We start with the free symmetric monoidal category F (G_0) whose objects are freely generated by a set G_0. We will think of G_0 interchangeably as a set or as a category whose objects are G_0 and which only has identity morphisms. Accordingly, we sometimes speak of functions out of G_0 and sometimes of functors.

Let C be an arbitrary symmetric monoidal category. Denote by $F^{G_0}(C)$ the category of functors from G_0 to C.Explicity, a functor $\Phi \in F^{G_0}(C)$ is just a function $G_0 \to Obj(C)$ since for morphisms there is no freedom-identity morphisms must be mapped to identity morphisms.

Given another symmetric monoidal category C′, we can look at symmetric monoidal functors Ψ: C→C′ that are compatible with a choice of $\Phi \in F^{G_0}(C)$ and $\Phi' \in F^{G_0}(C')$ in the sense that

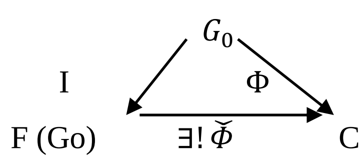


Commutes up to natural isomorphism. In this setup, we may ask if there is a symmetric category F (G_0) together with a functor I: $G_0 \rightarrow$ F (G_0) such that the pair (F (G_0), I) is universal in the following sense: for all symmetric monoidal categories C, precomposing with I induces an equivalence of categories

$$\boldsymbol{Fun}_{\otimes,\boldsymbol{sym}}(\boldsymbol{F}(\boldsymbol{G_0}),\boldsymbol{C}) \xrightarrow{(-)\boldsymbol{oI}} \boldsymbol{F}^{\boldsymbol{G0}}(\boldsymbol{C}) \qquad (21)$$

We call the pair (F (G_0), I) the free symmetric monoidal category generated by$\boldsymbol{G_0}$**.** An equivalent way of stating the universal property (21) is to impose the following two conditions on the pair (F (G_0), I), which have to hold for every symmetric monoidal category C:=

(i) We require that for each function $\Phi: G_0 \rightarrow Obj(C)$ there exists a unique-up-to-natural-monoidal-isomorphism symmetric monoidal functor$\widetilde{\Phi} = F(G_0) \rightarrow C$, such that



(22)

Commutes up to natural isomorphism.

ii)Let Φ, Ψ, $G_0 \rightarrow Obj(C)$ be two functions. Note that a natural transformation$\emptyset: \Phi \rightarrow \Psi$is given by a collection$(\emptyset_x)x \in G_0$ of morphisms $\emptyset x: \Phi(x) \rightarrow \Psi(x)$ in C with no further conditions imposed. We require that for each collection of morphisms $\big(\emptyset x: \Phi(x) \rightarrow \Psi(x)\big)x \in Go$ in C there exists a unique natural monoidal transformations $\breve{\emptyset}: \breve{\Phi} \rightarrow \breve{\Psi}$ such that $\emptyset x = \breve{\emptyset} I(x)$ for all x ∈ G_o.

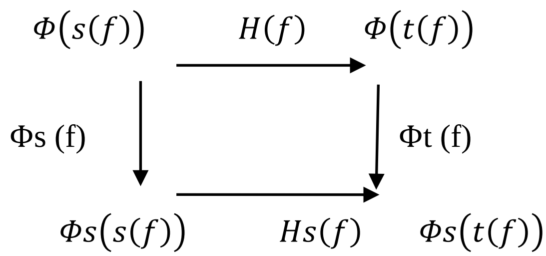
3.2 Objects and morphisms:

Having the symmetric monoidal category $F(G_0)$ at our disposal, we can try to add a set G_1 of extra morphisms to $F(G_0)$.more formally, pick a set G_1 and maps$s, t: G_1 \rightarrow Obj(F(Go))$.

Let C be an arbitrary symmetric monoidal category. We embellish the functor category $\boldsymbol{Fun}_{\otimes,\boldsymbol{sym}}(\boldsymbol{F}(\boldsymbol{G_0}),\boldsymbol{C})$ from above by including a choice of morphism for each element of G_1. That is, we define a category $F^{Go,G1}(C)$ with

1. Objects: pairs (Φ, H), where $\boldsymbol{\Phi} \in \boldsymbol{Fun}_{\otimes,\boldsymbol{sym}}(F(G_0), C)$ and his a map from G_1 into Mor(C) such that for $f \in G_1, H(f)$ has source $\Phi(s(f))$ and target $\Phi\big(t(f)\big)$.

2. Morphisms (Φ, H) →(Φ′ , H′): monoidal transformations $\emptyset: \Phi \rightarrow \Phi s$ which make



Commute for each $f \in G_1$.

The free symmetric monoidal category generated by Go and G_1 is the universal such category in the following sense: it is a symmetric monoidal category $F(G_o, G_1)$ together with a pair
(J, j) of a symmetric monoidal functor J: $F(G_0) \rightarrow F(G_o, G_1)$ and function j: $G_1 \rightarrow Mor(F(G_o, G_1))$ as above, such that for each symmetric monoidal category the functor

$$Fun_{\otimes,sym}(F(G_0, G_1), C) \rightarrow F^{Go,G1}(C),\ \Phi \rightarrow (\Phi oJ, \Phi oj) \quad (23)$$

is an equivalence of categories.
In the above example of one-dimensional topological quantum field theories, G_1 is as in (19) above, with s, t given by s ( ) = ( ), ( ) = ($\bullet_+$, $\bullet_-$) , etc.to give a symmetric monoidal functor $F(G_o, G_1)$→ Vectk amounts to picking U,V as in step 1,together with morphisms b: k → U⊗V and d:V⊗U → k.A monoidal natural transformation to another such functor with data (U′,V′,b′,d′) amounts to choosing linear maps f:U→U′, g:V→V′ such that
(f ⊗ g)ob = b′ and d′o (g ⊗ f) = d.Note that at this point.U,V do not have to be finite-dimensional. This will be enforced only by the relations we turn to next.

3.3 Objects, morphisms and relations:

We already have $F(G_o, G_1)$ at our disposal. Then the relations G_2 are a set of diagrams in $F(G_o, G_1)$ which we would like to commute. We formalize this by saying that an element of G_2 is a pair (f_1, f_2), where f_1, f_2: x→y are morphisms in $F(G_o, G_1)$.

Let again C be an arbitrary symmetric monoidal category. We define the category $F^{Go,G1,G2}(C)$ to be the full subcategory of $Fun_{\otimes,sym}(F(G_0, G_1), C)$ whose objects are those symmetric monoidal functors F with satisfy F (f_1) =F (f_2) for each pair (f_1, f_2)∈ G_2.

Definition: 3.1
A symmetric monoidal category freely generated by objects G_o, morphisms G_1, and relations G_2 is
(i) A symmetric monoidal category,
(ii)A symmetric monoidal functor S: $F(G_o, G_1)$→F such that S(f_1)=S(f_2) for each pair (f_1, f2) ∈ G_2.
Such that for each symmetric monoidal category C, the functor

$$\boldsymbol{Fun_{\otimes,sym}(F, C)} \xrightarrow{(-)os} \boldsymbol{F^{G0,G1,G2}(C)} \quad (24)$$

Is equivalence of categories. We denote such a category by $F(G_o, G_1, G_2)$:=F.

In the above example of one-dimensional topological quantum field theories, G_2 is the two-element set from (20) above. To give a symmetric monoidal functor $F(G_o, G_1, G_2)$→Vectk then amounts to picking(U,V,b,d) as above, but now subject to the relations

$$(d\otimes id)o(id\otimes b) = id, \qquad (id\otimes d)o(b\otimes id) = id. \tag{25}$$

We note that this precisely describes an object of DPk.We already saw that a natural monoidal transformation between two such functors is the same as giving morphism in DPk.we obtain an equivalence of categories

$$Fun_{\otimes,sym}(F(G_o, G_1, G_2), Vectk) \rightarrow DPk. \tag{26}$$

CONCLUSION

Hence conclude that there is one merit in the full version of construction, and that is that one is immediately led to the right notion of morphisms between Frobenius algebras. The economy version would suggest algebra homomorphisms which preserve the pairing as morphisms, but this would still allow injective maps which are not bijective, and we would not get the equivalence of this notion of Frobenius algebra morphisms.

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