

Generalized Differential Operators Involving Multivariable Hyper geometric Function

A. K. Thakur and Kiran Rajwade

Department of Mathematics,
Dr. C. V. Raman University, Bilaspur, INDIA.

(Received on: May 8, Accepted: May 10, 2017)

ABSTRACT

In this paper we use differential operators we to derive a formula of multivariable Hyper geometric function. In this generalization Leibnitz's rule for fractional derivative in order to obtain one the aforementioned formulas which involve a product of two multivariable's Hyper geometric function.

Keywords: Fractional differential operator, multivariable H-function, Leibnitz's rule.

1. INTRODUCTION AND DEFINITIONS

The fractional derivative of special function of one and more variables is important such as in the evaluation of series,^{10,15} the derivation of generating function^{12, chap.5} and the solution of differential equation^{4, 14, chap-3} motivated by these and many other avenues of application. The fractional differential operators $D_{k,\alpha,x}^n$ and ${}_a D_x^\mu$ are much used in the theory of special function of one and more variables.

We use the fractional derivative operator defined in the following manner¹⁶

$$D_{k,\alpha,x}^n(x^\mu) = \left[\frac{\sqrt{\mu+rk+1}}{\sqrt{\mu+rk-\alpha+1}} \right] x^{\mu+nk} \quad (1)$$

Where $\alpha \neq \mu + 1$ and α and K are not necessarily integers.

We use the binomial expansion in the following manner

$$(ax^\mu + b)^\lambda = b^\lambda \sum_{l=0}^{\infty} \binom{\lambda}{l} \left(\frac{ax^\mu}{b}\right)^l \quad \text{where } \left[\frac{ax^\mu}{b}\right] < 1 \quad (2)$$

The familiar differential operator ${}_a D_x^\mu$ is defined by⁵,

$${}_a D_x^\mu f(x) = \begin{cases} \frac{1}{\sqrt{-\mu}} \int_a^x (x-t)^{-\mu-1} f(t) dt, & [\text{Re}(\mu) < 0] \\ \frac{d^m}{dx^m} {}_a D_x^{\mu-m} f(x), & [0 \leq \text{Re}(\mu) < m] \end{cases} \quad (3)$$

Where m is a positive integer

For $\alpha = 0$ (3) Defines the classical Riemann- Liouville fractional derivative of order μ ($0 < \mu < 1$) when $\alpha \rightarrow \infty$ (3) may be identified with the definition of the well known Weyl fraction derivative of order μ ($0 < \mu < 1$)[1,chap.13];3] the special case of fractional calculus operator ${}_x D^\mu$ when $\alpha = 0$ is written as D_x^μ thus we have

$$D_x^\mu = {}_0 D_x^\mu \tag{4}$$

In this paper we derive several fractional derivative formulas involving multivariable H-function which as defined by Srivastav and Panda⁸ thus following the various conventions and notations explained fairly and fully in their earlier works⁶.

$$H[z_1, \dots, z_r] = H_{p, q; p_1 q_1, \dots, p_r q_r}^{0, n; m_1, n_1, \dots, m_r, n_r} \left(\begin{matrix} z_1 \\ \alpha_j, \alpha_j^{(1)}, \dots, \alpha_j^{(r)} \end{matrix} \middle| \begin{matrix} (c_j^1, \gamma_j^1)_{1, p_1}, \dots, (c_j^r, \gamma_j^r)_{1, p_r} \\ z_r \\ (b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1, q_1}, (d_j^1, \delta_j^1)_{1, q_1}, \dots, (d_j^r, \delta_j^r)_{1, q_r} \end{matrix} \right) \tag{5}$$

Denote the H- function of r -variables z_1, z_2, \dots, z_r here for convenience

$$(a_1, \alpha_1^1, \dots, \alpha_1^r)_{1, p} \text{ Abbreviates the P- member array } (\alpha_1^1, \dots, \alpha_1^r), \dots, (a_p, \alpha_p^1, \dots, \alpha_p^r) \tag{6}$$

While $(c_j^{(i)}, \gamma_j^{(i)})_{1, p_i}$ Abbreviates the array of p_i pairs of parameters

$$(\alpha_j^{(i)}, \gamma_j^{(i)}), \dots, (\alpha_{p_i}^{(i)}, \gamma_{p_i}^{(i)}); \quad (i = 1, \dots, r) \tag{7}$$

And so on, suppose, as usual that the parameters

$$\begin{aligned} a_j, \quad j = 1, \dots, p; \quad c_j^{(i)}, \quad j = 1, \dots, p_i \\ b_j, \quad j = 1, \dots, q; \quad d_j^{(i)}, \quad j = 1, \dots, q_i; \quad \forall i \in (i = 1, \dots, r) \end{aligned} \tag{8}$$

Are complex number and the associated coefficients

$$\begin{aligned} \alpha_j, \quad j = 1, \dots, p; \quad \gamma_j^{(i)}, \quad j = 1, \dots, p_i \\ \beta_j, \quad j = 1, \dots, q; \quad \delta_j^{(i)}, \quad j = 1, \dots, q_i; \quad \forall i \in (i = 1, \dots, r) \end{aligned} \tag{9}$$

Are positive real numbers such that

$$\Delta_i = \sum_{j=1}^r \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \leq 0 \tag{10}$$

And

$$\Omega_i = -\sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0; \quad \forall i \in (i = 1, \dots, r) \tag{11}$$

Where the integers $n, p, q, m_i, n_i, p_i, q_i$ are constrained by the inequalities $0 \leq n \leq P, q \geq 0,$

$1 \leq m_i \leq q_i, 0 \leq n_i \leq P_i [i = 1, \dots, r]$ and the equality (10) holds true for suitably restricted values of the complex variables z_1, \dots, z_r

Then it is known that the multiple Mellin-Barnes counter integral¹¹, representing the multivariable H- function (5) converges absolutely under the condition (11) when

$$|\arg(z_i)| < \frac{1}{2} \Delta \Omega_i, \quad \forall i \in (i = 1, \dots, r) \quad (12)$$

$$H[z_1, \dots, z_r] = \begin{cases} 0(|z_1|^{\xi_1} \dots |z_r|^{\xi_r}), & (\max |z_1| \dots |z_r| \rightarrow 0) \\ 0(|z_1|^{\eta_1} \dots |z_r|^{\eta_r}), & (\eta = 0; \min |z_1| \dots |z_r| \rightarrow \infty) \end{cases} \quad (13)$$

Where with $i=1, \dots, r$ min

$$\begin{aligned} \xi_i &= \min \left\{ \operatorname{Re} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right\}, & (j=1, \dots, m_i) \\ \eta_i &= \max \left\{ \operatorname{Re} \left(c_j^{(i)} - \frac{1}{\gamma_j^{(i)}} \right) \right\}, & (j=1, \dots, n_i) \end{aligned} \quad (14)$$

Provided that each of the inequalities holds true.

Throughout the present paper we assume that the convergence and existence condition corresponding appropriately to the ones detained above are satisfied by each of the various H-function involved in our results which are presented in the following sections.

2. MAIN RESULT

In this section we shall prove our main formula on fractional differential operator involving multivariable H- function.

Result (1)

$$\begin{aligned} & D_{k, \alpha, x}^n \{ x^l (x^{v_1} + a)^\lambda (b - x^{v_2})^{-\delta} \\ & H [z_1 x^{\rho_1} (x^{v_1} + a)^{\sigma_1} (b - x^{v_2})^{-\delta_1}, \dots, z_r x^{\rho_r} (x^{v_1} + a)^{\sigma_r} (b - x^{v_2})^{-\delta_r}] \} \\ & = a^\lambda b^{-\delta} x^{l+nk} \sum_{l, m=0}^{\infty} \frac{(x^{v_1}/a)^l (x^{v_2}/b)^m}{l! m!} H \quad \begin{matrix} 0, n + n + 2 : m_1, n_1, \dots, m_r, n_r \\ p + n + 2, q + n + 2 : p_1, q_1 ; \dots ; p_r, q_r \end{matrix} \\ & \left[\begin{array}{l} z_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\rho_r} a^{\sigma_r} b^{-\delta_r} \end{array} \right] \left(\begin{array}{l} (-\lambda, \sigma_1, \dots, \sigma_r), (1 - \delta - m, \delta_1, \dots, \delta_r), (-t - gk - v_1 l - v_2 m; \rho_1, \dots, \rho_r)_{g=0, n-1}, (\alpha_j, \alpha_j^{(i)}, \dots, \alpha_j^{(r)})_{1, p} \\ (c_j^{(1)}, \gamma_j^{(1)})_{1, p_1}, \dots, (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \\ (-\lambda + l, \sigma_1, \dots, \sigma_r), (1 - \delta, \delta_1, \dots, \delta_r), (\alpha - t - gk - v_1 l - v_2 m; \rho_1, \dots, \rho_r)_{g=0, n-1}, (b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1, q} \\ (d_j^{(1)}, \delta_j^{(1)})_{1, q_1}, \dots, (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{array} \right) \end{aligned}$$

Provided (in addition to appropriate convergence and existing condition) that

$$\begin{aligned} & \min \{v_1, v_2, \rho_i, \sigma_i, \delta_i\} > 0 \quad (i = 1, \dots, r) ; \\ & \max \{ |\arg(x^{v_1}/a)|, |\arg(x^{v_2}/b)| \} < \Pi \end{aligned}$$

Where $\xi_i = (i = 1, \dots, r)$.
 $\text{Re}(k) + \prod_{i=1}^r k_i \xi_i > -1$

Proof:

$$D_{k,\alpha,x}^n \{x^i (x^{u_1} + a)^{\lambda_1} (b - x^{u_2})^{-\delta_1} (x^{u_3} + c)^{\lambda_2} (d - x^{u_4})^{-\delta_2} \\
H [z_1 x^{\rho_1} (x^{u_1} + a)^{\sigma_1} (b - x^{u_2})^{-\delta_1} (x^{u_3} + c)^{\sigma_2} (d - x^{u_4})^{-\delta_2}, \dots, z_r x^{\rho_r} (x^{u_1} + a)^{\sigma_r} (b - x^{u_2})^{-\delta_r} \\
(x^{u_3} + c)^{\sigma_r} (d - x^{u_4})^{-\delta_r}]\} \\
= a^{\lambda_1} b^{-\delta_1} c^{\lambda_2} d^{-\delta_2} x^{t+nk} \sum_{k,l,m,n=0}^{\infty} \frac{(x^{u_1}/a)^k (x^{u_2}/b)^l (x^{u_3}/c)^m (x^{u_4}/d)^n}{k! l! m! n!} H \\
0, n+n+4: k_1 l_1 m_1 n_1, \dots, k_r l_r m_r n_r \\
p+n+4, q+n+4: p_1, q_1, \dots, p_r, q_r$$

$$\left[\begin{array}{l} z_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} c^{\sigma_2} d^{-\delta_2} \\ \vdots \\ z_r x^{\rho_r} a^{\sigma_r} b^{-\delta_r} c^{\sigma_r} d^{-\delta_r} \end{array} \right] \left(\begin{array}{l} (-\lambda, \sigma_1, \dots, \sigma_r), (1 - \delta - k - m, \delta_1, \dots, \delta_r), (-t - gk - v_1 k - v_2 l - v_3 m - v_4 n; \rho_1, \dots, \rho_r)_{g=0, n-1}, (\alpha_j, \alpha_j^{(i)}, \dots, \alpha_j^{(r)})_{1,p} \\ : (c_j^{(1)}, y_j^{(1)})_{1,p_1}, \dots, (c_j^{(r)}, y_j^{(r)})_{1,p_r} \\ (-\lambda + 1, \sigma_1, \dots, \sigma_r), (1 - \delta, \delta_1, \dots, \delta_r), (\alpha - t - gk - v_1 k - v_2 l - v_3 m - v_4 n; \rho_1, \dots, \rho_r)_{g=0, n-1}, (b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} \\ : (d_j^{(1)}, \delta_j^{(1)})_{1,q_1}, \dots, (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right)$$

Provided (in addition to the appropriate convergence and existing condition) that

$$\min \{u_1, u_2, v_3, u_4, \rho_i, \sigma_i, \delta_i\} > 0 \quad (i = 1, \dots, r) ; \\
\max \{|\arg(x^{u_1}/a)|, |\arg(x^{u_2}/b)|, |\arg(x^{u_3}/c)|, |\arg(x^{u_4}/d)|\} < \Pi \\
\text{Re}(k) + \prod_{i=1}^r k_i \xi_i > -1 \quad \text{Re}(k) + \prod_{i=1}^r k_i \xi_i > -1 \\
\text{Where } \xi_i = (i = 1, \dots, r) .$$

Result (2)

$$D_{k,\alpha,x}^n D_y^\mu \{x^i y^\lambda (x^{u_1} + a)^\lambda (b - x^{u_2})^{-\delta} (y^{u_3} + c)^h (d - y^{u_4})^{-g} \\
H [z_1 x^{\rho_1} y^{\lambda_1} (x^{u_1} + a)^{\sigma_1} (b - x^{u_2})^{-\delta_1} (y^{u_3} + c)^{h_1} (d - y^{u_4})^{-g_1}, \dots, z_r x^{\rho_r} y^{\lambda_r} (x^{u_1} + a)^{\sigma_r} (b - \\
x^{u_2})^{-\delta_r} (y^{u_3} + c)^{h_r} (d - y^{u_4})^{-g_r}]\} \\
= a^\lambda b^{-\delta} c^h d^{-g} x^{t+nk} y^{\lambda-\mu} \sum_{l,m,r,s=0}^{\infty} \frac{(x^{u_1}/a)^l (x^{u_2}/b)^m (x^{u_3}/c)^r (x^{u_4}/d)^s}{l! m! r! s!} H \\
0, n + n + 5: m_1 n_1, \dots, m_r n_r \\
p + n + 5, q + n + 5: p_1, q_1, \dots, p_r, q_r$$

$$\left[\begin{array}{l} z_1 x^{\rho_1} y^{\lambda_1} a^{\sigma_1} b^{-\delta_1} c^{h_1} d^{-g_1} \\ \vdots \\ z_r x^{\rho_r} y^{\lambda_r} a^{\sigma_r} b^{-\delta_r} c^{h_r} d^{-g_r} \end{array} \right] \left(\begin{array}{l} (-\lambda, \sigma_1, \dots, \sigma_r), (1 - \delta - m, \delta_1, \dots, \delta_r), (-h, h_1, \dots, h_r), (1 - g - t, g_1, \dots, g_r) (-\lambda - rv_3 - sv_4; \lambda_1, \dots, \lambda_r) \\ (-t - gk - v_1 l - v_2 m; k_1, \dots, k_r)_{g=0, n-1}, (\alpha_j, \alpha_j^{(i)}, \dots, \alpha_j^{(r)})_{1,p}: (c_j^{(1)}, y_j^{(1)})_{1,p_1}, \dots, (c_j^{(r)}, y_j^{(r)})_{1,p_r} \\ (-\lambda + 1, \sigma_1, \dots, \sigma_r), (1 - \delta, \delta_1, \dots, \delta_r), (-h + r, h_1, \dots, h_r), (1 - g, g_1, \dots, g_r) (-\lambda + \mu - rv_3 - sv_4; \lambda_1, \dots, \lambda_r) \\ (\alpha - t - gk - v_1 l - v_2 m; k_1, \dots, k_r)_{g=0, n-1}, (b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q}: (d_j^{(1)}, \delta_j^{(1)})_{1,q_1}, \dots, (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right)$$

provided (in addition to the appropriate convergence and existing condition) that

$$\min \{u_1, u_2, v_3, u_4, \rho_i, \sigma_i, \delta_i, h_i, g_i\} > 0 \quad (i = 1, \dots, r) ;$$

$$\max \{ |\arg(x^{u_1}/a)|, |\arg(x^{u_2}/b)|, |\arg(y^{v_3}/c)|, |\arg(y^{v_4}/d)| \} < \Pi$$

$$\operatorname{Re}(k) + \prod_{i=1}^r k_i \xi_i > -1 \quad \operatorname{Re}(\lambda) + \prod_{i=1}^r \lambda_i \xi_i > -1$$

Where $\xi_i = (i = 1, \dots, r)$ are given in (14)

$$D_{k,\alpha,x}^n D_y^\mu D_w^\eta \{ x^t y^\lambda w^\gamma (x^{u_1} + a)^\lambda (b - x^{u_2})^{-\delta} (y^{v_3} + c)^h (d - y^{v_4})^{-g} (w^{v_5} + e)^\theta (f - w^{v_6})^{-\varphi}$$

$$H [z_1 x^{\rho_1} y^{\lambda_1} w^{\gamma_1} (x^{u_1} + a)^{\sigma_1} (b - x^{u_2})^{-\delta_1} (y^{v_3} + c)^{h_1} (d - y^{v_4})^{-g_1} (w^{v_5} + e)^{\theta_1} (f - w^{v_6})^{-\varphi_1}, \dots, z_r x^{\rho_r} y^{\lambda_r} w^{\gamma_r} (x^{u_1} + a)^{\sigma_r} (b - x^{u_2})^{-\delta_r} (y^{v_3} + c)^{h_r} (d - y^{v_4})^{-g_r} (w^{v_5} + e)^{\theta_r} (f - w^{v_6})^{-\varphi_r}] \}$$

$$= a^\lambda b^{-\delta} c^h d^{-g} e^\theta f^{-\varphi} x^{t+nk}$$

$$y^{\lambda-\mu} w^{\gamma-\eta} \sum_{k,l,m,n,r,s=0}^{\infty} \frac{(x^{u_1}/a)^k (x^{u_2}/b)^l (y^{v_3}/c)^m (y^{v_4}/d)^n (w^{v_5}/e)^r (w^{v_6}/f)^s}{k! l! m! n! r! s!} H$$

$$\begin{matrix} 0, n+n+7: m_1 n_1, \dots, m_r n_r \\ p+n+7, q+n+7: p_1, q_1, \dots, p_r, q_r \end{matrix}$$

$$\left[\begin{matrix} z_1 x^{\rho_1} y^{\lambda_1} a^{\sigma_1} b^{-\delta_1} c^{h_1} d^{-g_1} e^{\theta_1} f^{-\varphi_1} \\ \vdots \\ z_r x^{\rho_r} y^{\lambda_r} a^{\sigma_r} b^{-\delta_r} c^{h_r} d^{-g_r} e^{\theta_r} f^{-\varphi_r} \end{matrix} \right] \left(\begin{matrix} (-\sigma, \sigma_1, \dots, \sigma_r), (1 - \delta - m, \delta_1, \dots, \delta_r), (-h, h_1, \dots, h_r), (1 - g - t, g_1, \dots, g_r), (-\theta, \theta_1, \dots, \theta_r) (-\lambda - rv_3 - sv_4 - \theta v_5 - \varphi v_6 \\ ; \lambda_1, \dots, \lambda_r) (-t - gk - v_1 l - v_2 m; k_1, \dots, k_r)_{g=0, n-1}, (\alpha_j, \alpha_j^{(i)}, \dots, \alpha_j^{(r)})_{1, p}; (c_j^{(1)}, \gamma_j^{(1)})_{1, p_1}, \dots, (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \\ (-\sigma + l, \sigma_1, \dots, \sigma_r), (1 - \delta, \delta_1, \dots, \delta_r), (-h + r, h_1, \dots, h_r), (1 - g, g_1, \dots, g_r) (-\theta + r, \theta_1, \dots, \theta_r) (-\lambda + \mu - rv_3 - sv_4 - \theta v_5 - \varphi v_6 \\ ; \lambda_1, \dots, \lambda_r) (\alpha - t - gk - v_1 l - v_2 m; k_1, \dots, k_r)_{g=0, n-1}, (b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1, q}; (d_j^{(1)}, \delta_j^{(1)})_{1, q_1}, \dots, (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{matrix} \right)$$

Provided (in addition to the appropriate convergence and existing condition) that

$$\min \{v_1, v_2, v_3, v_4, v_5, v_6, \rho_i, \sigma_i, \delta_i, h_i, g_i, \theta_i, \varphi_i\} > 0 \quad (i = 1, \dots, r) ;$$

$$\max \{ |\arg(x^{u_1}/a)|, |\arg(x^{u_2}/b)|, |\arg(y^{v_3}/c)|, |\arg(y^{v_4}/d)|, |\arg(w^{v_5}/e)|, |\arg(w^{v_6}/f)| \} < \Pi$$

$$\operatorname{Re}(k) + \prod_{i=1}^r k_i \xi_i > -1 \quad \operatorname{Re}(\lambda) + \prod_{i=1}^r \lambda_i \xi_i > -1 \quad \operatorname{Re}(\gamma) + \prod_{i=1}^r \gamma_i \xi_i > -1$$

Where $\xi_i = (i = 1, \dots, r)$.

REFERENCES

1. Eredlyi A. *et al.*, tables of integral transform, vol.2 mc graw hill, NY/Toronto/ London (1954).
2. Lavoie J. I, osler and tremblay R: Fractional derivatives and special Functions SIAM, Rev 18, 240-268 (1976).
3. Samko, S.G. Kilbas, A. A. and Maricev, O.L.: integrals and Derivatives of Fractional order some of their applications, Nauka, Tekhnika minsk, in Russian (1987).
4. Nishimoto, K: fractional Calculus Vol.1-4 Descrates Press, Koriyama, (1984, 1987, 1989 and 1991).
5. Oldham K.B. and spanier.j: the Fractional Calculus, Acadmic Press Ny/London, (1974).
6. Srivastava, H.M. and Panda R: some expansion theorem for the H-Function of several complex variables I and II, *Comment, Math, Univ. St. Paul*, 24, fasc.2, 119-137; *ibid*25 (1976),Fasc.2,167- 197 (1975).

7. Srivastava H.M. and panda R: Some bilateral generations Function of several complex variables. *J. Reine Angew Math.* 288, 129-145 (1976).
8. Srivastav, H.M. and panda R: Some bilateral generations Function for a class of generalized hypergeometric polynomials *J. Reine Angew Math.* 283-284, 265-275 (1976).
9. Srivastava, H.m and Monocha, H.L.: A Treatise on generating Functions Halsted Press Chichester and wiley. NY/Chichester/Brisbane/Toronto (1984).
10. Srivastava, H.m and Goyal S.P.: Fractional Derivatives of the H-Function of several variables, *J. Math. Anal Appl.* 112, 2, 641-651 (1985).